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ON THE STABILITY OF NONCONSERVATIVE SYSTEMS WITH SMALL DISSIPATION

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Abstract. In the present work, we study the paradoxical influence of small dissipative and gyroscopic forces on the stability of linear nonconservative systems consisting of the nonpredictable (at first glance) behavior of a critical nonconservative loading. By studying bifurcations of multiple roots of the characteristic polynomial of the nonconservative system considered, the analytical description of this effect is obtained. The model of a disk brake describing the appearance of a creak in the braking of a car is considered as a mechanical example.

1. Introduction

1. Let us consider the following linear, autonomous, nonconservative mechanical system:

$$\frac{d^2\boldsymbol{y}}{dt^2} + \boldsymbol{D}(\boldsymbol{k})\frac{d\boldsymbol{y}}{dt} + \boldsymbol{A}(q)\boldsymbol{y} = 0, \tag{1.1}$$

where $\boldsymbol{y}$ is the vector of generalized coordinates and $\boldsymbol{D}$ and $\boldsymbol{A}$ are real square matrices of order m; the first of them defines the dissipative and gyroscopic forces, and the second defines the nonconservative positional forces. It is assumed that the matrix $\boldsymbol{D}$ is a smooth function of the parameter vector $\boldsymbol{k} = (k_1, \ldots, k_{n-1})$, and, moreover, $\boldsymbol{D}(0) = 0$, whereas the matrix $\boldsymbol{A}$ smoothly depends on the scalar loading parameter $q \geq 0$.

Seeking a solution of Eq. (1.1) having the form $\boldsymbol{y} = \boldsymbol{u}\exp(\lambda t)$, we arrive at the following generalized eigenvalue problem:

$$(\lambda^2\boldsymbol{I} + \lambda\boldsymbol{D}(\boldsymbol{k}) + \boldsymbol{A}(q))\boldsymbol{u} = 0, \tag{1.2}$$

where $\boldsymbol{I}$ is the identity matrix of order m, $\boldsymbol{u}$ is an eigenvector, and λ is the corresponding eigenvalue. The nonconservative system

$$\frac{d^2\boldsymbol{y}}{dt^2} + \boldsymbol{A}(q)\boldsymbol{y} = 0 \tag{1.3}$$

without gyroscopic and dissipative forces ($\boldsymbol{k} = 0$) is said to be *circulation* system [4, 17]. The spectrum of the circulation system is symmetric with respect to the real and imaginary axes of the complex plane: if λ is an eigenvalue of the linear operator $\lambda^2\boldsymbol{I} + \boldsymbol{A}(q)$, then $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$, where the bar over a symbol means the complex conjugation, are also its eigenvalues. As a consequence, a circulation system is Lyapunov stable if and only if all eigenvalues λ are purely imaginary and semisimple [13].

Assume that for $q = 0$, the circulation system (1.3) is stable. When the loading parameter grows, its eigenvalues moves along the imaginary axis. In this process, two simple, purely imaginary eigenvalues can compose a double eigenvalue $\lambda_0 = i\omega_0$ with Jordan chain of length 2 when the parameter q attains a certain critical value q_0. In general, the further growth of the loading leads to the fact that λ_0 falls into two simple, complex eigenvalues, one of which has a positive real part, which means oscillatory instability (flutter; see Fig. 1(a)). Therefore, the interval $0 \leq q < q_0$ belongs to the stability region of the circulation system (1.3).

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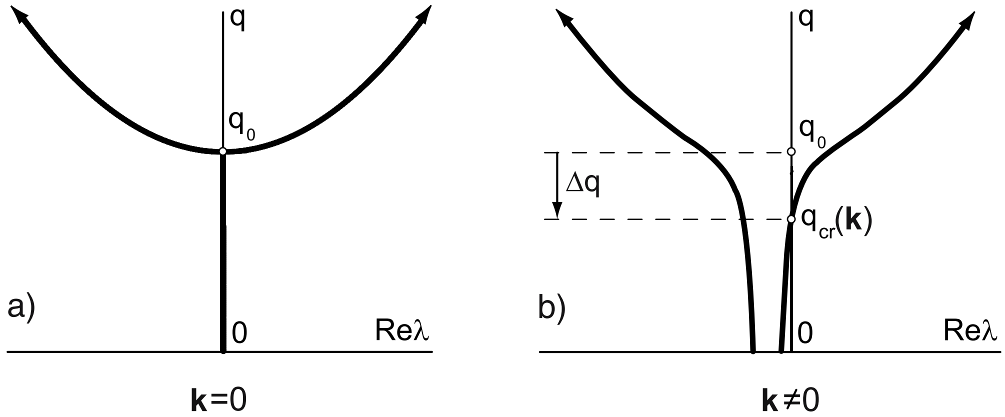


Fig. 1. Trajectories of eigenvalues demonstrating the destabilization paradox.

As was discovered in studying various mechanical problems [4–6, 10, 11, 13, 14, 16, 17], in the general case a perturbation of a circulation system by small dissipative and gyroscopic forces ($\boldsymbol{k} \neq 0$) destroys the interaction of eigenvalues and leads to the fact that when the loading parameter attains a certain critical value $q = q_{\text{cr}}(k_1, \ldots, k_{n-1})$, one of the eigenvalues passes to the right part of the complex plane without formation of a double eigenvalue and its subsequent bifurcation (see Fig. 1(b)). Moreover, it turns out that if we set $\boldsymbol{k} = \epsilon\widetilde{\boldsymbol{k}}$, where the vector $\widetilde{\boldsymbol{k}}$ is fixed, and let the parameter ϵ tend to zero, then

$$\widetilde{q}_{\text{cr}} \equiv \lim_{\epsilon \to 0} q_{\text{cr}}(\epsilon\widetilde{\boldsymbol{k}}) \leq q_0. \tag{1.4}$$

An inequality of the form (1.4) was first obtained by Ziegler in 1952 [17]; studying the stability of a two-link pendulum loaded by a tracing force, he arrived at the surprising conclusion that the critical force of the stability loss for a nonconservative system with a vanishing small dissipation is substantially less than in the case where it is assumed that there is no dissipation in the system from the very beginning. This phenomenon was called the *destabilization paradox* [4–6, 10, 11, 13, 14, 16, 17].

Later on, examining various mechanical systems, it was shown that the limit of the critical loading $\widetilde{q}_{\text{cr}}$ depends on the choice of the vector $\widetilde{\boldsymbol{k}}$. In particular, changing the relation between the parameters $k_1, \ldots, k_{n-1}$, we can avoid the reduction of the critical loading and, therefore, the destabilization (Bolotin's effect [4]). In [8, 12], the destabilization paradox was studied for general nonconservative systems of the form (1.1). Explicit analytical expressions were found for the critical loading q; they depend on the parameter vector $\boldsymbol{k}$ and describe the jump of the critical loading. Also, the behavior of eigenvalues of a nonconservative system with a small friction was analytically described. In [12], the authors found the structure of the dissipative and gyroscopic force matrix stabilizing the circulation system considered and also necessary and sufficient stability conditions. The results of [8, 12] were obtained by using the perturbation theory of matrix eigenvalues [15] and the eigenvectors and the associated vectors of the circulation system (1.3).

In the present work, we study the destabilization paradox based on the sensitivity analysis for roots of the characteristic polynomial of system (1.1) whose coefficients are found by using the modified Leverrier algorithm [2, 7]. This allows us to avoid the calculation of eigenvectors and associated vectors and to obtain expressions for the critical loading through the derivatives of the characteristic polynomial with respect to the parameters and, in the case of a system with two degrees of freedom, directly through the invariants of the matrices $\boldsymbol{D}$ and $\boldsymbol{A}$.

2. Bifurcations of Roots of the Characteristic Polynomial

The influence of small dissipative and gyroscopic forces on the stability of the nonconservative system (1.1) that is on the boundary between the stability and flutter regions is stabilizing or nonstabilizing depending on the way in which the double eigenvalue splits under the change of the parameters q and $\boldsymbol{k}$ and on the behavior of the simple eigenvalues in this case.

The eigenvalues of system (1.1) are roots of the characteristic polynomial

$$P(\lambda, \boldsymbol{p}) = \lambda^{2m} + \sum_{s=1}^{2m} a_s \lambda^{2m-s} = \det(\boldsymbol{I}\lambda^2 + \boldsymbol{D}(\boldsymbol{k})\lambda + \boldsymbol{A}(q)). \tag{2.1}$$

The coefficients of polynomial (2.1) are smooth functions of the vector $\boldsymbol{p} = (k_1, k_2, \ldots, k_{n-1}, q)$ of n real parameters and are expressed through the invariants of the matrices $\boldsymbol{D}$ and $\boldsymbol{A}$ via the Leverrier algorithm [2, 7]:

$$\begin{gathered} \boldsymbol{C}_0 = \boldsymbol{I}, \quad a_1 = \operatorname{tr} \boldsymbol{D}(\boldsymbol{k}), \quad \boldsymbol{C}_1 = -\boldsymbol{D}(\boldsymbol{k}) + a_1 \boldsymbol{I}, \\ a_j = \frac{1}{j} \operatorname{tr}(\boldsymbol{D}(\boldsymbol{k})\boldsymbol{C}_{j-1} + 2\boldsymbol{A}(q)\boldsymbol{C}_{j-2}), \quad \boldsymbol{C}_j = -\boldsymbol{D}(\boldsymbol{k})\boldsymbol{C}_{j-1} - \boldsymbol{A}(q)\boldsymbol{C}_{j-2} + a_j \boldsymbol{I}, \\ j = 2, 3, \ldots, 2m-2, \\ \boldsymbol{D}(\boldsymbol{k})\boldsymbol{C}_{2m-2} + \boldsymbol{A}(q)\boldsymbol{C}_{2m-3} = a_{2m-1}\boldsymbol{I}, \quad \boldsymbol{A}(q)\boldsymbol{C}_{2m-2} = a_{2m}\boldsymbol{I}. \end{gathered} \tag{2.2}$$

For $\boldsymbol{p} = \boldsymbol{p}_0$, let the function $P(\lambda, \boldsymbol{p}_0)$ have a root $\lambda = \lambda_0$. Let us study the behavior of this root under a variation of the parameter vector $\boldsymbol{p}$ along a smooth curve:

$$\boldsymbol{p}(\epsilon) = \boldsymbol{p}_0 + \epsilon \dot{\boldsymbol{p}} + \frac{\epsilon^2}{2} \ddot{\boldsymbol{p}} + o(\epsilon^2), \quad \epsilon \geq 0, \tag{2.3}$$

where the dot denotes the differentiation in the parameter ϵ, and, moreover, the derivatives are calculated for $\epsilon = 0$. Then the polynomial $P(\lambda, \boldsymbol{p}(\epsilon))$ can be represented in the form

$$P(\lambda, \boldsymbol{p}(\epsilon)) = \sum_{r=0}^{2m} \frac{(\lambda - \lambda_0)^r}{r!} \left(\frac{\partial^r P}{\partial \lambda^r} + \epsilon \frac{\partial^r P_1}{\partial \lambda^r} + \epsilon^2 \frac{\partial^r P_2}{\partial \lambda^r} + o(\epsilon^2) \right), \tag{2.4}$$

where

$$\frac{\partial^r P_1}{\partial \lambda^r} = \sum_{s=1}^{n} \frac{\partial^{r+1} P}{\partial \lambda^r \partial p_s} \dot{p}_s, \quad \frac{\partial^r P_2}{\partial \lambda^r} = \frac{1}{2} \sum_{s=1}^{n} \frac{\partial^{r+1} P}{\partial \lambda^r \partial p_s} \ddot{p}_s + \frac{1}{2} \sum_{s,t=1}^{n} \frac{\partial^{r+2} P}{\partial \lambda^r \partial p_s \partial p_t} \dot{p}_s \dot{p}_t, \tag{2.5}$$

and the partial derivatives are calculated for $\boldsymbol{p} = \boldsymbol{p}_0$, $\lambda = \lambda_0$. For $r = 0$, formulas (2.5) yield the expressions for the quantities P_1 and P_2.

Let $\lambda_0 = i\omega_0$; then the collapse process of the double root λ_0 of the characteristic polynomial (2.1) under perturbation (2.3) is described by the Newton–Puiseux series [15]

$$\lambda = \lambda_0 + \lambda_1 \epsilon^{1/2} + \lambda_2 \epsilon + \lambda_3 \epsilon^{3/2} + \lambda_4 \epsilon^2 + \ldots. \tag{2.6}$$

Substituting expansions (2.4) and (2.6) in the equation $P(\lambda, \boldsymbol{p}) = 0$ and collecting the coefficients of by the same powers of ϵ, we obtain the following relations defining the coefficients of expansion (2.6):

$$P(\lambda_0, \boldsymbol{p}_0) = 0, \tag{2.7}$$

$$\lambda_1 \frac{\partial P}{\partial \lambda}\bigg|_{\substack{\lambda=\lambda_0 \\ \boldsymbol{p}=\boldsymbol{p}_0}} = 0, \tag{2.8}$$

$$\left(P_1 + \frac{1}{2}\lambda_1^2 \frac{\partial^2 P}{\partial \lambda^2} + \lambda_2 \frac{\partial P}{\partial \lambda} \right)\bigg|_{\substack{\lambda=\lambda_0 \\ \boldsymbol{p}=\boldsymbol{p}_0}} = 0, \tag{2.9}$$

$$\left(\lambda_1\lambda_2\frac{\partial^2 P}{\partial\lambda^2}+\lambda_3\frac{\partial P}{\partial\lambda}+\lambda_1\frac{\partial P_1}{\partial\lambda}+\lambda_1^3\frac{1}{6}\frac{\partial^3 P}{\partial\lambda^3}\right)\bigg|_{\substack{\lambda=\lambda_0\\ \boldsymbol{p}=\boldsymbol{p}_0}}=0. \tag{2.10}$$

Equations (2.7) and (2.8) hold automatically if λ_0 is a double root of the characteristic polynomial (2.1). Using Eq. (2.8), we find from Eqs. (2.9) and (2.10) the coefficients λ_1 and λ_2 in expansion (2.6):

$$\lambda_1^2=-P_1\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1},\quad \lambda_2=\left(\frac{1}{3}\frac{\partial^3 P}{\partial\lambda^3}P_1-\frac{\partial P_1}{\partial\lambda}\frac{\partial^2 P}{\partial\lambda^2}\right)\left(\frac{\partial^2 P}{\partial\lambda^2}\right)^{-2}, \tag{2.11}$$

where all the derivatives are calculated for $\boldsymbol{p}=\boldsymbol{p}_0$, $\lambda=\lambda_0$.

Therefore, under a variation of parameters of the form (2.3), the double eigenvalue $\lambda_0=i\omega_0$ with Jordan chain of length 2 breaks down according to the formula

$$\lambda=i\omega_0\pm i\sqrt{\epsilon P_1\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}} \\ -\frac{\epsilon}{2}\left(\frac{\partial P_1}{\partial\lambda}\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}-\frac{1}{3!}\frac{\partial^3 P}{\partial\lambda^3}\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}P_1\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}\right)+o(\epsilon) \tag{2.12}$$

if the expression under the square root in (2.12) does not vanish for $\epsilon\neq 0$ [14].

The case $P_1=0$ is degenerate, since, according to Eqs. (2.11), the coefficient λ_1 in expansion (2.6) vanishes. Substituting expansions (2.4) and (2.6) in the equation $P(\lambda,\boldsymbol{p})$ with account for the condition $\lambda_1=0$ and collecting the coefficients of the same power of ϵ, we find that

$$\left(P_1+\lambda_2\frac{\partial P}{\partial\lambda}\right)\bigg|_{\substack{\lambda=\lambda_0\\ \boldsymbol{p}=\boldsymbol{p}_0}}=0, \tag{2.13}$$

$$\lambda_3\frac{\partial P}{\partial\lambda}\bigg|_{\substack{\lambda=\lambda_0\\ \boldsymbol{p}=\boldsymbol{p}_0}}=0, \tag{2.14}$$

$$\left(P_2+\lambda_2^2\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}+\lambda_2\frac{\partial P_1}{\partial\lambda}+\lambda_4\frac{\partial P}{\partial\lambda}\right)\bigg|_{\substack{\lambda=\lambda_0\\ \boldsymbol{p}=\boldsymbol{p}_0}}=0. \tag{2.15}$$

By the existence condition of a double root of $\partial P/\partial\lambda=0$ and by the degeneracy condition $P_1=0$ at the point $\boldsymbol{p}=\boldsymbol{p}_0$, Eqs. (2.13) and (2.14) hold automatically. Equation (2.15) yields a polynomial of the second degree that serves for finding the coefficient λ_2 in expansion (2.6):

$$\lambda_2^2\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}+\lambda_2\frac{\partial P_1}{\partial\lambda}+P_2=0, \tag{2.16}$$

where all the derivatives are calculated for $\lambda=\lambda_0$ and $\boldsymbol{p}=\boldsymbol{p}_0$. Therefore, in the degenerate case where $P_1=0$, the double root λ_0 breaks down according to the formula $\lambda=\lambda_0+\lambda_2\epsilon+o(\epsilon)$, where the coefficient λ_2 is found from Eq. (2.16).

Analogously, we can show that the behavior of simple eigenvalues $\lambda_{0,s}$ is described by the following formula under the variation of parameters (2.3) (see [14]):

$$\lambda=\lambda_{0,s}+\mu_1\epsilon+o(\epsilon),\quad \mu_1=-P_1\left(\frac{\partial P}{\partial\lambda}\right)^{-1}\bigg|_{\substack{\lambda=\lambda_{0,s}\\ \boldsymbol{p}=\boldsymbol{p}_0}}. \tag{2.17}$$

Therefore, the asymptotic formulas (2.6), (2.12), (2.16), and (2.17) whose coefficients are expressed through the derivatives of the characteristic polynomial describe the behavior of the simple and double eigenvalues under the change of parameters in the regular and degenerate cases.

3. Stability Analysis of a Nonconservative System

In the n-dimensional space of parameters $k_1, \ldots, k_{n-1}$, q of system (1.1), let us consider the point $\boldsymbol{p}_0 = (0, \ldots, 0, q_0)$ assuming that $\pm\lambda_0 = \pm i\omega_0$, $\omega_0 > 0$, are double eigenvalues of the operator $\boldsymbol{A}(q_0) + \lambda^2\boldsymbol{I}$ with Jordan chain of length 2 and the other eigenvalues $\pm\lambda_{0,s} = \pm i\omega_{0,s}$, $\omega_{0,s} > 0$, $s = 1, \ldots, m-2$, are purely imaginary and simple. The nonconservative system corresponding to $\boldsymbol{k} = 0$, $q = q_0$, is a circulation system, and the point $\boldsymbol{p}_0$ belongs to the stability region [4, 13].

It follows from Eqs. (2.2) that the following relations hold for the odd coefficients of the characteristic polynomial $a_{2r-1}(\boldsymbol{p})$ and the matrices $\boldsymbol{C}_{2j-1}(\boldsymbol{p})$:

$$a_{2r-1}(\boldsymbol{p}_0) = 0, \quad \boldsymbol{C}_{2j-1}(\boldsymbol{p}_0) = 0, \quad r = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, m-1, \tag{3.1}$$

where $\boldsymbol{p}_0 = (0, \ldots, 0, q_0)$. On the other hand, using expressions (2.2), by induction we can show that

$$\begin{aligned} &\frac{\partial a_{2r}}{\partial k_s}(\boldsymbol{p}_0) = 0, \qquad \frac{\partial a_{2r-1}}{\partial q}(\boldsymbol{p}_0) = 0, \qquad r = 1, 2, \ldots, m, \\ &\frac{\partial \boldsymbol{C}_{2j}}{\partial k_s}(\boldsymbol{p}_0) = 0, \qquad \frac{\partial \boldsymbol{C}_{2j-1}}{\partial q}(\boldsymbol{p}_0) = 0, \qquad j = 1, 2, \ldots, m-1. \end{aligned} \tag{3.2}$$

Now let us consider formula (2.17) describing the behavior of the simple eigenvalue $\lambda_{0,s} = i\omega_{0,s}$ under the parameter variation. Define the scalar quantity $\widetilde{g}_s$ and the vector $\boldsymbol{g}_s = (g_1^s, g_2^s, \ldots, g_{n-1}^s)$ as follows:

$$\widetilde{g}_s = -i\frac{\partial P}{\partial q}\left(\frac{\partial P}{\partial \lambda}\right)^{-1}\Bigg|_{\substack{\lambda = i\omega_{0,s} \\ \boldsymbol{p}=\boldsymbol{p}_0}}, \quad g_j^s = \frac{1}{\omega_{0,s}}\frac{\partial P}{\partial k_j}\left(\frac{\partial P}{\partial \lambda}\right)^{-1}\Bigg|_{\substack{\lambda = i\omega_{0,s} \\ \boldsymbol{p}=\boldsymbol{p}_0}}. \tag{3.3}$$

By relations (3.1) and (3.2), the quantities $\widetilde{g}_s$ and g_j^s are real. With account for relations (2.3) and (3.3), expressions (2.17) take the form

$$\lambda = i\omega_{0,s} - i\widetilde{g}_s(q - q_0) - \omega_{0,s}\langle \boldsymbol{g}_s, \boldsymbol{k}\rangle + \ldots, \tag{3.4}$$

where the corner brackets denote the inner product of real vectors: $\langle \boldsymbol{a}, \boldsymbol{b}\rangle = \sum\limits_{j=1}^{n-1} a_j b_j$.

Therefore, a small change of the loading parameter q leaves the simple eigenvalues $\lambda_{0,s} = i\omega_{0,s}$ on the imaginary axis, whereas the dissipative and gyroscopic forces represented by the parameter vector $\boldsymbol{k}$ translate $\lambda_{0,s}$ to the left or right half-planes of the complex plane. The latter occurs under the condition

$$\langle \boldsymbol{g}_s, \boldsymbol{k}\rangle > 0, \quad s = 1, 2, \ldots, m-2. \tag{3.5}$$

The collapse of the double eigenvalue $\lambda_0 = i\omega_0$ in the nondegenerate case is described by formula (2.12). Define the scalar quantity $\widetilde{f}$ and the vector $\boldsymbol{f} = (f_1, f_2, \ldots, f_{n-1})$ by the relations

$$\widetilde{f} = \frac{\partial P}{\partial q}\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda = i\omega_0 \\ \boldsymbol{p}=\boldsymbol{p}_0}}, \quad f_s = \frac{1}{i\omega_0}\frac{\partial P}{\partial k_s}\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda = i\omega_0 \\ \boldsymbol{p}=\boldsymbol{p}_0}}. \tag{3.6}$$

By relations (3.1) and (3.2), quantities (3.6) are real. On the other hand, let us introduce into consideration the vector $\boldsymbol{h} = (h_1, h_2, \ldots, h_{n-1})$ and the quantity $\tilde{h}$, which, in view of relations (3.1) and (3.2) are real:

$$\begin{aligned} h_s &= \frac{1}{\omega_0}\left(\frac{1}{2!}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\left(\frac{2\alpha_0}{i\omega_0}\frac{\partial P}{\partial k_s} - \frac{\partial^2 P}{\partial \lambda \partial k_s}\right)\Bigg|_{\substack{\lambda = i\omega_0 \\ \boldsymbol{p}=\boldsymbol{p}_0}}, \\ \widetilde{h} &= \left(\frac{1}{2!}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}\left(\frac{\alpha_0}{\omega_0}\frac{\partial P}{\partial q} - i\frac{\partial^2 P}{\partial \lambda \partial q}\right)\Bigg|_{\substack{\lambda = i\omega_0 \\ \boldsymbol{p}=\boldsymbol{p}_0}}, \\ \alpha_0 &= i\omega_0\frac{1}{3!}\frac{\partial^3 P}{\partial \lambda^3}\left(\frac{1}{2!}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}. \end{aligned} \tag{3.7}$$

Taking into account the quantities defined by expressions (3.6) and (3.7), we write Eq. (2.12) in the form

$$\lambda = i\omega_0 \pm i\sqrt{\epsilon i\omega_0\langle \boldsymbol{f}, \dot{\boldsymbol{k}}\rangle + \epsilon\widetilde{f}\dot{q}} - \epsilon\frac{1}{2}(\langle \alpha_0\boldsymbol{f} - \omega_0\boldsymbol{h}, \dot{\boldsymbol{k}}\rangle + i\widetilde{h}\dot{q}) + o(\epsilon). \tag{3.8}$$

It follows from Eq. (3.8) that in the generic case, the double eigenvalue $\lambda_0 = i\omega_0$ splits into two complex-conjugated values, one of which has a positive real part (oscillatory instability or flutter). However, under the conditions $\langle \boldsymbol{f}, \dot{\boldsymbol{k}}\rangle = 0$, $\widetilde{f}\dot{q} > 0$, and $\langle \boldsymbol{h}, \dot{\boldsymbol{k}}\rangle < 0$, the double eigenvalue falls into two simple eigenvalues with negative real parts (asymptotic stability). Taking into account that, in the first approximation, $\boldsymbol{k} = \epsilon\dot{\boldsymbol{k}}$ and $q = q_0 + \epsilon\dot{q}$ and also assuming that $\widetilde{f} < 0$, we write these conditions in the form

$$\langle \boldsymbol{f}, \boldsymbol{k}\rangle = 0, \quad q < q_0, \quad \langle \boldsymbol{h}, \boldsymbol{k}\rangle < 0. \tag{3.9}$$

In the parameter space, conditions (3.5) and (3.9) define the set of directions directing from the point $\boldsymbol{p}_0$ to the asymptotic stability region, i.e., the tangent cone to this region at the point $\boldsymbol{p}_0$. The tangent cone (3.5), (3.9) is degenerate, since its dimension $n-1$ is less than the dimension n of the stability region [1, 14].

To obtain a more detailed approximation of the asymptotic stability region, let us consider the behavior of eigenvalues under variations of parameters along curves (2.3) tangent to set (3.9) and orthogonal to the axis q. The following conditions must hold for this purpose:

$$\langle \boldsymbol{f}, \dot{\boldsymbol{k}}\rangle = 0, \quad \dot{q} = 0, \tag{3.10}$$

which means that the expression under the square root in formula (3.8) vanishes. Under such a degeneration, the collapse of the double eigenvalue λ_0 is described by expansion (2.6) with $\lambda_1 = 0$ and with the coefficient λ_2 found from Eq. (2.16).

Taking into account the quantities defined by expressions (3.6) and (3.7), under conditions (3.10) we write Eq. (2.16) in the form

$$\lambda_2^2 - \omega_0\langle \boldsymbol{h}, \dot{\boldsymbol{k}}\rangle\lambda_2 + \frac{1}{2}\widetilde{f}\ddot{q} + \omega_0^2\langle \boldsymbol{G}\dot{\boldsymbol{k}}, \dot{\boldsymbol{k}}\rangle + i\omega_0\left(\frac{1}{2}\langle \boldsymbol{f}, \ddot{\boldsymbol{k}}\rangle + \langle \boldsymbol{H}\dot{\boldsymbol{k}}, \dot{\boldsymbol{k}}\rangle\right) = 0, \tag{3.11}$$

where the entries of real matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are defined by the relations

$$H_{st} = \frac{1}{2\omega_0}\operatorname{Im}\left(\frac{\partial^2 P}{\partial k_s\partial k_t}\right)\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_0\\ \boldsymbol{p}=\boldsymbol{p}_0}}, \tag{3.12}$$

$$G_{st} = \frac{1}{2\omega_0^2}\operatorname{Re}\left(\frac{\partial^2 P}{\partial k_s\partial k_t}\right)\left(\frac{1}{2}\frac{\partial^2 P}{\partial\lambda^2}\right)^{-1}\Bigg|_{\substack{\lambda=i\omega_0\\ \boldsymbol{p}=\boldsymbol{p}_0}}, \quad s, t = 1, 2, \ldots, n-1. \tag{3.13}$$

The roots of polynomial (3.11) with complex coefficients lie in the left part of the complex plane if the conditions of the Bilharz criterion [3] hold. Taking into account expressions (2.3) and (3.10), we write these conditions in the form

$$q < q_{\mathrm{cr}}(\boldsymbol{k}), \quad \langle \boldsymbol{h}, \boldsymbol{k}\rangle < 0, \tag{3.14}$$

$$q_{\mathrm{cr}}(\boldsymbol{k}) = q_0 + \frac{(\langle \boldsymbol{f}, \boldsymbol{k}\rangle + \langle \boldsymbol{H}\boldsymbol{k}, \boldsymbol{k}\rangle)^2}{\widetilde{f}\langle \boldsymbol{h}, \boldsymbol{k}\rangle^2} - \frac{\omega_0^2}{\widetilde{f}}\langle \boldsymbol{G}\boldsymbol{k}, \boldsymbol{k}\rangle. \tag{3.15}$$

Together with inequalities (3.5), conditions (3.14) approximate the asymptotic stability region of the nonconservative system (1.1) in a neighborhood of the point $\boldsymbol{p}_0 = (0, \ldots, 0, q_0)$. As often happens, if we assume that under (3.14), conditions (3.5) also hold, then Eq. (3.15) also approximates the boundary of the asymptotic stability region in a neighborhood of the point $\boldsymbol{p}_0$.

In the important particular case where the nonconservative system (1.1) has only two degrees of freedom ($m = 2$), there are no stability conditions (3.5) corresponding to the simple eigenvalues, and all the things depend on the collapse of the double eigenvalue λ_0. It follows from relations (2.1) and (2.2) that the characteristic polynomial of the system with two degrees of freedom has the form

$$P = \lambda^4 + \lambda^3 \operatorname{tr} \boldsymbol{D}(\boldsymbol{k}) + \lambda^2(\operatorname{tr} \boldsymbol{A}(q) + \det \boldsymbol{D}(\boldsymbol{k})) + \lambda(\operatorname{tr} \boldsymbol{A}(q) \operatorname{tr} \boldsymbol{D}(\boldsymbol{k}) - \operatorname{tr}(\boldsymbol{A}(q)\boldsymbol{D}(\boldsymbol{k}))) + \det \boldsymbol{A}(q). \quad (3.16)$$

When there are no dissipative and gyroscopic forces ($\boldsymbol{k} = 0$), the solutions of Eq. (3.16) are written in explicit form:

$$\lambda = \pm\sqrt{-\frac{\operatorname{tr} \boldsymbol{A}(q)}{2} \pm \frac{1}{2}\sqrt{(\operatorname{tr} \boldsymbol{A}(q))^2 - 4 \det \boldsymbol{A}(q)}}. \quad (3.17)$$

For the critical values of the loading parameter $q = q_0$, the expression $(\operatorname{tr} \boldsymbol{A}(q))^2 - 4 \det \boldsymbol{A}(q)$ under the square root in (3.11) vanishes, which leads to the formation of the pair of double complex-conjugate eigenvalues

$$\pm\lambda_0 = \pm i\omega_0, \quad \omega_0 = \sqrt{\frac{\operatorname{tr} \boldsymbol{A}(q_0)}{2}} > 0. \quad (3.18)$$

With account for the dissipative and gyroscopic forces ($\boldsymbol{k} \neq 0$), the collapse of these double roots leads to the stability conditions (3.14).

In the case of $m = 2$ degrees of freedom, the quantities $\widetilde{f}$ and $\widetilde{h}$ and the components of the vectors $\boldsymbol{f}$ and $\boldsymbol{h}$ entering relations (3.8), (3.14), and (3.15) are directly expressed through the invariants of the matrices $\boldsymbol{A}$ and $\boldsymbol{D}$:

$$\begin{aligned} \widetilde{f} &= \frac{1}{4\omega_0^2} \operatorname{tr}\left((\boldsymbol{A}(q_0) - \omega_0^2 \boldsymbol{I})\frac{\partial \boldsymbol{A}}{\partial q}\right), \qquad & f_s &= \frac{1}{4\omega_0^2} \operatorname{tr}\left((\boldsymbol{A}(q_0) - \omega_0^2 \boldsymbol{I})\frac{\partial \boldsymbol{D}}{\partial k_s}\right), \\ \widetilde{h} &= \frac{1}{4\omega_0^3} \operatorname{tr}\left((\boldsymbol{A}(q_0) - 3\omega_0^2 \boldsymbol{I})\frac{\partial \boldsymbol{A}}{\partial q}\right), \qquad & h_s &= \frac{1}{4\omega_0^3} \operatorname{tr}\left((\boldsymbol{A}(q_0) - 3\omega_0^2 \boldsymbol{I})\frac{\partial \boldsymbol{D}}{\partial k_s}\right). \end{aligned} \quad (3.19)$$

The components of the real matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ are found by the formulas

$$H_{st} = \frac{1}{8\omega_0^2} \operatorname{tr}\left((\boldsymbol{A}_0 - \omega_0^2 \boldsymbol{I})\frac{\partial^2 \boldsymbol{D}}{\partial k_s \partial k_t}\right), \quad G_{st} = \frac{1}{8\omega_0^2}\left(\operatorname{tr}\frac{\partial \boldsymbol{D}}{\partial k_s} \operatorname{tr}\frac{\partial \boldsymbol{D}}{\partial k_t} - \operatorname{tr}\left(\frac{\partial \boldsymbol{D}}{\partial k_s}\frac{\partial \boldsymbol{D}}{\partial k_t}\right)\right), \quad (3.20)$$

where $s, t = 1, 2, \ldots, n-1$. The derivatives in relations (3.19) and (3.20) are calculated for $\boldsymbol{k} = 0$ and $q = q_0$.

In conclusion, we note that Eq. (3.11) implies the relations describing the trajectories of eigenvalues on the complex plane and also the expression for real and imaginary parts of eigenvalues depending on the parameters:

$$(\operatorname{Im} \lambda - \omega_0 - \operatorname{Re} \lambda - a/2)^2 - (\operatorname{Im} \lambda - \omega_0 + \operatorname{Re} \lambda + a/2)^2 = 2d, \quad (3.21)$$

$$(\operatorname{Re} \lambda + a/2)^4 + \left(c - a^2/4\right)(\operatorname{Re} \lambda + a/2)^2 = d^2/4, \quad (3.22)$$

$$(\operatorname{Im} \lambda - \omega_0)^4 - \left(c - a^2/4\right)(\operatorname{Im} \lambda - \omega_0)^2 = d^2/4. \quad (3.23)$$

The coefficients a, c, and d of Eqs. (3.21)–(3.23) have the form

$$a = -\omega_0\langle \boldsymbol{h}, \boldsymbol{k}\rangle, \quad c = \widetilde{f}(q - q_0) + \omega_0^2\langle \boldsymbol{G}\boldsymbol{k}, \boldsymbol{k}\rangle, \quad d = \omega_0(\langle \boldsymbol{f}, \boldsymbol{k}\rangle + \langle \boldsymbol{H}\boldsymbol{k}, \boldsymbol{k}\rangle). \quad (3.24)$$

4. Example. Two-Dimensional Model of a Disk Brake

The compression of a rotating flexible disk of thickness $2h$ from two opposite sides can lead to its transversal vibrations with frequencies lying in the sound range (creak of brakes). In [9], Popp et al. described a simple model with two degrees of freedom describing the mechanism of flutter appearance for a rotating disk under its braking. Let us consider a transversal section of an element of the disk in the plane perpendicular to its radius (Fig. 2). It is assumed that an element of the disk of mass m with moment of inertia J can turn around its center of mass by an angle φ, whereas the center of mass can displace in the vertical direction by a quantity x with respect to the equilibrium state. The viscoelastic

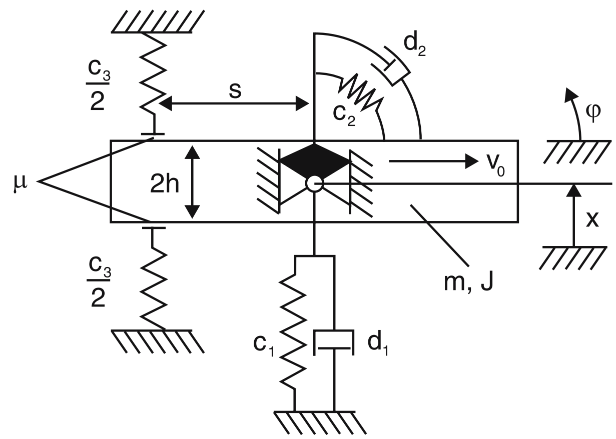


Fig. 2. Element of the rotating disk with contact points on the upper and lower surfaces [9].

properties of the disk are modeled by regenerating forces and momenta with rigidity coefficients c_1 and c_2 and viscosity coefficients d_1 and d_2, which influence the translational and rotational motions of the element, respectively (Fig. 2). The disk is subject to the action of two brake shoes with contact rigidities $c_3/2$ lying on the upper and lower surfaces of the disk distant by s from the center of rotation of the element considered. The Coulomb friction at the contact points produces a resistance force proportional to the normal pressure force with coefficient μ. It is assumed that when there is no loading, the element of the disk moves with a constant velocity v_0. Smooth vibrations of the system in a neighborhood of the equilibrium state arising under the action of the brake shoes on the disk are described by Eq. (1.1), where $\boldsymbol{y} = [x, \varphi]^T$ and the matrices $\boldsymbol{D}$ and $\boldsymbol{A}$ have the following form [9]:

$$\boldsymbol{D} = \frac{1}{mJ}\begin{bmatrix} Jd_1 & 0 \\ 0 & md_2 \end{bmatrix}, \quad \boldsymbol{A} = \frac{1}{mJ}\begin{bmatrix} J(c_1+c_3) & -Jc_3 s \\ m(\mu h - s)c_3 & m(s-\mu h)c_3 s + mc_2 \end{bmatrix}. \tag{4.1}$$

First, let us consider the system without damping ($d_1 = 0$, $d_2 = 0$). Solving the equation $(\operatorname{tr}\boldsymbol{A})^2 = 4\det\boldsymbol{A}$ for such a system, we find the following critical value of the friction coefficient μ whose excess causes the vibration of the disk:

$$\mu_0 = \frac{s}{h} + \frac{mc_2 + J(c_3 - c_1) - 2\sqrt{c_3 J(mc_2 - Jc_1)}}{hsmc_3}. \tag{4.2}$$

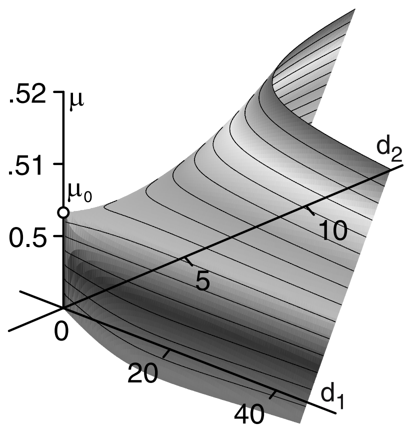
When μ approaches the critical value (4.2), two eigenfrequencies approach one another and compose the double critical frequency ω_0. Taking into account (4.2), we find from the equation $\omega_0^2 = \operatorname{tr}\boldsymbol{A}(\mu_0)/2$ that

$$\omega_0^2 = \frac{c_1}{m} + \frac{\sqrt{c_3 J(mc_2 - Jc_1)}}{Jm}. \tag{4.3}$$

The further growth of the friction coefficient causes the collapse of the double eigenvalue $i\omega_0$ into two simple eigenvalues, one of which goes to the right half-plane of the complex plane leading to vibrations of the disk, which manifest themselves in the form of the disk creak.

For the following values of the parameters presented in [9]:

$$\begin{gathered} m = 50\,\mathrm{kg}, \quad J = 10\,\mathrm{kg}\cdot\mathrm{m}^2, \\ c_1 = 10\,\mathrm{N}\cdot\mathrm{m}^{-1}, \quad c_2 = 10\,\mathrm{N}\cdot\mathrm{m}, \quad c_3 = 60\,\mathrm{N}\cdot\mathrm{m}^{-1}, \\ s = 1\,\mathrm{m}, \quad h = 2\,\mathrm{m}, \end{gathered} \tag{4.4}$$



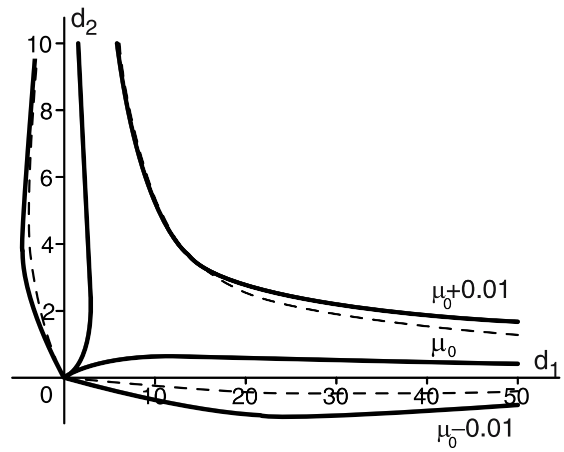


Fig. 3. Critical value of the friction coefficient μ as a function of d_1 and d_2 and its level lines for the values of the parameters from (4.4).

the critical friction coefficient and the frequency are equal to

$$\mu_0 = \frac{2}{3} - \frac{1}{15}\sqrt{6} \simeq 0.50337, \tag{4.5}$$

$$\omega_0 = \frac{1}{5}\sqrt{5 + 10\sqrt{6}} \simeq 1.08618\,\mathrm{s}^{-1}\,. \tag{4.6}$$

The critical value of the friction coefficient $\mu_{\rm cr}$ for the system with damping ($d_1 \neq 0$, $d_2 \neq 0$) is found by using the Routh–Hurwitz criterion applied to the characteristic polynomial of the system (1.1), (4.1):

$$\mu_{\rm cr} = \frac{s}{h} + \frac{mc_2 - Jc_1}{mhsc_3} + \frac{Jd_1^2 + md_1d_2}{2m^2hsc_3} + \frac{J^2d_1^2 + m^2d_2^2}{2m^2hsd_1d_2} - \frac{Jd_1 + md_2}{2d_1d_2m^2hsc_3}\Big[(c_3(Jd_1 - md_2) + d_2d_1^2)^2 - 4md_1d_2(c_1c_3J - mc_2c_3 + d_1d_2c_1)\Big]^{1/2}. \tag{4.7}$$

Surface (4.7) and its level lines are shown in Fig. 3. For the values of the parameters taken from (4.4) and for $d_1 = 1\,\mathrm{N\cdot s\cdot m^{-1}}$ and $d_2 = 1\,\mathrm{N\cdot m\cdot s}$, using (4.7) we find that (see [9])

$$\mu_{\rm cr} = \frac{24803}{30000} - \frac{\sqrt{10553201}}{10000} \simeq 0.50191 < \mu_0 \simeq 0.50337. \tag{4.8}$$

Inequality (4.8) expresses the destabilization paradox by small dissipative forces: the critical value of the friction coefficient reduces by a jump. Substituting (4.4) and (4.8) in the characteristic equation of the system (1.1), (4.1), we find the critical frequency corresponding to the critical friction coefficient (4.8):

$$\omega_{\rm cr} \simeq 1.15304\,\mathrm{s}^{-1}\,. \tag{4.9}$$

Comparing (4.6) and (4.9), we can conclude that the critical frequency of vibration of the disk also changes by a jump whenever the dissipative forces are taken into account.

Now let us find the approximations of critical values of the friction coefficient and the frequency using the general formulas of the present paper. Substituting matrices (4.1) in expressions (3.19) and (3.20), we find the quantity $\widetilde{f}$ and the matrix $\boldsymbol{G}$:

$$\widetilde{f} = \frac{c_3sh(J(\omega_0^2m - c_1) - 2\sqrt{c_3J(mc_2 - Jc_1)})}{4mJ^2\omega_0^2}, \quad \boldsymbol{G} = \frac{1}{8\omega_0^2mJ}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{4.10}$$

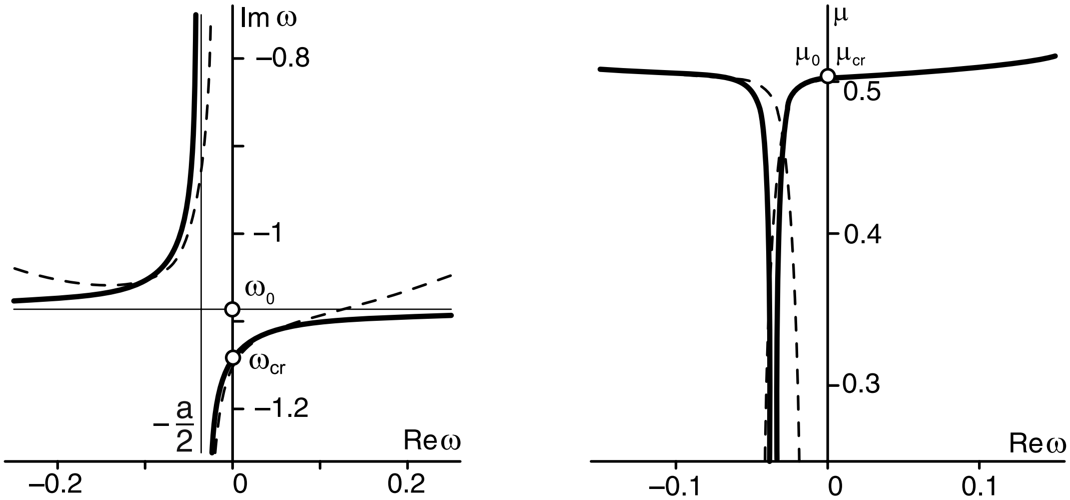


Fig. 4. Trajectories of eigenvalues and their approximations for $d_1 = 1\,\mathrm{N\cdot s\cdot m^{-1}}$ and $d_2 = 1\,\mathrm{N\cdot m\cdot s}$.

and also the vectors $\boldsymbol{f}$ and $\boldsymbol{h}$:

$$\boldsymbol{f} = \left(\frac{c_1 + c_3 - \omega_0^2 m}{4\omega_0^2 m^2},\ \frac{J(c_1 - c_3 - \omega_0^2 m) + 2\sqrt{c_3 J(mc_2 - Jc_1)}}{4\omega_0^2 m J^2} \right), \tag{4.11}$$

$$\boldsymbol{h} = \left(\frac{c_1 + c_3 - 3\omega_0^2 m}{4\omega_0^3 m^2},\ \frac{J(c_1 - c_3 - 3\omega_0^2 m) + 2\sqrt{c_3 J(mc_2 - Jc_1)}}{4\omega_0^3 m J^2} \right). \tag{4.12}$$

Using (4.11) and (4.12) in Eq. (3.15) and taking (4.4) into account, we find the critical friction coefficient in the form

$$\mu_{\mathrm{cr}}(d_1, d_2) = \mu_0 - \frac{1153\sqrt{6} - 2448}{3000} \frac{(d_1 - 5d_2)^2}{(5d_1 + (13 + 7\sqrt{6})d_2)^2} + \frac{12 + \sqrt{6}}{72000} d_1 d_2. \tag{4.13}$$

The surface of the critical friction coefficient (4.7) and its approximation (4.13) are shown in Fig. 3.

It follows from (4.13) that for $d_1 = 5d_2$ there is no jump of the critical friction coefficient. For the damping coefficients $d_1 = 1\,\mathrm{N\cdot s\cdot m^{-1}}$, we obtain the following for $d_2 = 1\,\mathrm{N\cdot m\cdot s}$ using (4.13):

$$\mu_{\mathrm{cr}} = \frac{119777}{6000} - \frac{63559}{8000}\sqrt{6} \simeq 0.50194. \tag{4.14}$$

Moreover, the stability-loss frequency found by the approximate formula (3.21) is equal to

$$\omega_{\mathrm{cr}} = \frac{14\sqrt{6} - 29}{25}\sqrt{5 + 10\sqrt{6}} \simeq 1.14980\,\mathrm{s}^{-1}. \tag{4.15}$$

The comparison of expressions (4.8), (4.9), (4.14), and (4.15) shows that the approximations of the critical quantities obtained from studying the collapse of multiple roots of the characteristic polynomial yield a good approximation of the exact solutions.

Using (4.10)–(4.12) in formulas (3.21)–(3.23), we can find the approximation of trajectories of eigenvalues on the complex plane near $i\omega_0$. For the values of the parameters given in (4.4), the quantities defined by Eqs. (3.24) have the form

$$a = \frac{38 - 7\sqrt{6}}{2300} d_1 + \frac{43 + 6\sqrt{6}}{920} d_2, \quad c = \frac{6(\sqrt{6} - 12)}{23}(\mu - \mu_0) + \frac{d_1 d_2}{2000},$$
$$d = \left(-\frac{33 + 3\sqrt{6}}{2300} d_2 + \frac{-15 + 7\sqrt{6}}{11500} d_1 \right) \sqrt{5 + 10\sqrt{6}}. \tag{4.16}$$

The trajectories of eigenvalues calculated in [9] by solving the characteristic equation for the values of the parameters given in (4.4) for $d_1 = 1\,\mathrm{N}\cdot\mathrm{s}\cdot\mathrm{m}^{-1}$ and $d_2 = 1\,\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}$, and for their approximation (3.21), (3.22), (4.16) are shown in Fig. 4 by the dotted and continuous lines, respectively.

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