Available online at www.sciencedirect.com
ScienceDirect
International Journal of Non-Linear Mechanics 42 (2007) 71–87
INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS
www.elsevier.com/locate/nlm

Destabilization paradox due to breaking the Hamiltonian and reversible symmetry

Oleg N. Kirillov[a,b,*]

[a]Dynamics group, Department of Mechanical Engineering, Technical University of Darmstadt, Hochschulstr. 1, 64289 Darmstadt, Germany
[b]Institute of Mechanics, Moscow State Lomonosov University, Michurinskii pr. 1, 119192 Moscow, Russia

Received 30 June 2006; received in revised form 5 September 2006; accepted 29 September 2006

Dedicated to the memory of Karl Popp

Abstract

Stability of a linear autonomous non-conservative system in the presence of potential, gyroscopic, dissipative, and non-conservative positional forces is studied. The cases when the non-conservative system is close to a gyroscopic system or to a circulatory one are examined. It is known that marginal stability of gyroscopic and circulatory systems can be destroyed or improved up to asymptotic stability due to action of small non-conservative positional and velocity-dependent forces. The present paper shows that in both cases the boundary of the asymptotic stability domain of the perturbed system possesses singularities such as "Dihedral angle" and "Whitney umbrella" that govern stabilization and destabilization. In case of two degrees of freedom, approximations of the stability boundary near the singularities are found in terms of the invariants of matrices of the system. As an example, the asymptotic stability domain of the modified Maxwell–Bloch equations is investigated with an application to the stability problems of gyroscopic systems with stationary and rotating damping.
© 2007 Elsevier Ltd. All rights reserved.

Keywords: Non-conservative system; Dissipation-induced instabilities; Destabilization paradox

1. Introduction

Consider an autonomous non-conservative system described by a linear differential equation of second order

$$\ddot{\mathbf{x}} + (\Omega\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + (\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0, \tag{1}$$

where dot denotes time differentiation, $\mathbf{x} \in \mathbb{R}^m$, and real matrix $\mathbf{K}=\mathbf{K}^T$ corresponds to potential forces. Real matrices $\mathbf{D}=\mathbf{D}^T$, $\mathbf{G} = -\mathbf{G}^T$, and $\mathbf{N} = -\mathbf{N}^T$ are related to dissipative (damping), gyroscopic, and non-conservative positional (circulatory) forces with magnitudes controlled by scaling factors δ, Ω, and ν, respectively.

General non-conservative system (1) has two important limiting cases corresponding to *circulatory* and *gyroscopic* systems. A circulatory system is obtained from (1) by neglecting velocity-dependent forces

$$\ddot{\mathbf{x}} + (\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0, \tag{2}$$

while a gyroscopic one has no damping and non-conservative positional forces

$$\ddot{\mathbf{x}} + \Omega\mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0. \tag{3}$$

Circulatory and gyroscopic systems (2) and (3) possess fundamental symmetries that are easily seen after transformation of Eq. (1) to the Cauchy form $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ with

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2}\Omega\mathbf{G} & \mathbf{I} \\ \frac{1}{2}\delta\Omega\mathbf{D}\mathbf{G} + \frac{1}{4}\Omega^2\mathbf{G}^2 - \mathbf{K} - \nu\mathbf{N} & \delta\mathbf{D} - \frac{1}{2}\Omega\mathbf{G} \end{pmatrix},$$

$$\mathbf{y} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} + \frac{1}{2}\Omega\mathbf{G}\mathbf{x} \end{pmatrix}, \tag{4}$$

where $\mathbf{I}$ is the identity matrix.

* Corresponding author at: Institute of Mechanics, Moscow State Lomonosov University, Michurinskii pr. 1, 119192 Moscow, Russia.
E-mail addresses: kirillov@dyn.tu-darmstadt.de, kirillov@imec.msu.ru (O.N. Kirillov).

0020-7462/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2006.09.003

Indeed, in the absence of damping and gyroscopic forces ($\delta = \Omega = 0$) the matrix $\mathbf{A}$ changes as $\mathbf{RAR} = -\mathbf{A}$ due to a coordinate transformation with the matrix

$$\mathbf{R} = \mathbf{R}^{-1} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}. \tag{5}$$

This means that the matrix $\mathbf{A}$ has a *reversible* symmetry, and Eq. (2) describes a reversible dynamical system [1,2]. Due to this property,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{R}(\mathbf{A} - \lambda\mathbf{I})\mathbf{R}) = \det(\mathbf{A} + \lambda\mathbf{I}), \tag{6}$$

and the eigenvalues of circulatory system (2) appear in pairs $(-\lambda, \lambda)$. Consequently, the equilibrium of a circulatory system is either unstable or all its eigenvalues lie on the imaginary axis of the complex plane implying marginal stability if they are semi-simple.

Without damping and non-conservative positional forces ($\delta = \nu = 0$) the matrix $\mathbf{A}$ possesses the *Hamiltonian* symmetry $\mathbf{JAJ} = \mathbf{A}^{\mathrm{T}}$, where $\mathbf{J}$ is a unit symplectic matrix [3]

$$\mathbf{J} = -\mathbf{J}^{-1} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}. \tag{7}$$

As a consequence,

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= \det(\mathbf{J}(\mathbf{A} - \lambda\mathbf{I})\mathbf{J}) = \det(\mathbf{A}^{\mathrm{T}} + \lambda\mathbf{I}) \\ &= \det(\mathbf{A} + \lambda\mathbf{I}),\end{aligned} \tag{8}$$

which implies that if λ is an eigenvalue of $\mathbf{A}$ then so is $-\lambda$, similarly to the reversible case. Therefore, gyroscopic system (3) can be only marginally stable with its spectrum belonging to the imaginary axis of the complex plane.

In the presence of all the four forces the Hamiltonian and reversible symmetries are broken and the marginal stability is generally destroyed. Instead, system (1) can be asymptotically stable if its characteristic polynomial

$$P(\lambda) = \det(\mathbf{I}\lambda^2 + (\Omega\mathbf{G} + \delta\mathbf{D})\lambda + \mathbf{K} + \nu\mathbf{N}) \tag{9}$$

satisfies the criterion of Routh and Hurwitz. The most interesting for many applications is the situation when system (1) is close either to circulatory system (2) with $\delta, \Omega \ll \nu$ (imperfect reversible system) or to gyroscopic system (3) with $\delta, \nu \ll \Omega$ (imperfect Hamiltonian system). Furthermore, the effect of small damping and gyroscopic forces on the stability of circulatory systems as well as the effect of small damping and non-conservative positional forces on the stability of gyroscopic systems are regarded as *paradoxical*, since the stability properties are extremely sensitive to the choice of the perturbation, and the balance of forces resulting in the asymptotic stability is not evident [4–48]. This characterization sounds even more justified if to take into account the connection of the *destabilization paradox* with the physical paradoxes such as “tippe top inversion” and “rising egg phenomenon” [36,37,44,47].

Historically, the destabilization paradox appeared first in a study of a gyroscopic system with dissipation by Thomson and Tait, who found that the dissipative perturbation destroys the gyroscopic stabilization so that the system is neither marginally nor asymptotically stable [4]. The terminology *dissipation-induced instabilities* has its roots in that classical work [20,47]. A similar effect of non-conservative positional forces on the stability of gyroscopic systems has been established almost a hundred years later by Lakhadanov and Karapetyan [11,12]. These ideas have been extensively developed, e.g., in the works [10,16,18–24,36,37,44–48,51].

A more sophisticated form of the destabilization paradox has been discovered by Ziegler on the example of a double pendulum loaded by a follower force with the damping non-uniformly distributed among the natural modes [7]. Without dissipation, the Ziegler pendulum is a circulatory system and it is marginally stable for the loads non-exceeding some critical value. Small dissipation makes the pendulum either unstable or asymptotically stable with the critical load, which can be significantly lower than that of the undamped system. This is caused by the singular nature of the new critical load, which is non-differentiable at the origin function of the damping parameters, having no limit when the damping coefficients uniformly tend to zero [8,9,19,26,51]. Numerous other aspects of the destabilization paradox by small velocity-dependent forces in circulatory systems, including more general settings and non-linear effects, have been investigated, e.g., in [8,13–15,17,25,26,28,33,35,38–43].

The destabilization paradox in Ziegler’s form has been revealed recently by Crandall in his study of a gyroscopic pendulum with stationary and rotating damping [22]. Contrary to the Ziegler pendulum, the undamped gyropendulum is a gyroscopic system that is marginally stable when its spin exceeds a critical value. Stationary damping corresponding to dissipative velocity-dependent force destroys the gyroscopic stabilization in accordance with the theorem of Thomson and Tait [4]. However, the Crandall gyropendulum with stationary and rotating damping, where the latter is related to non-conservative positional force, can be asymptotically stable for the rotation rates exceeding considerably the critical spin of the undamped system.

The growing number of other physical and mechanical examples demonstrating the destabilization paradox due to an interplay of non-conservative effects, requires a unified treatment of this phenomenon taking into account all types of forces presented in Eq. (1), as reported recently by Krechetnikov and Marsden [47].

The goal of the present paper is to find and analyze the domain of asymptotic stability of system (1) in the space of the parameters δ, Ω, and ν with special attention to imperfect reversible and Hamiltonian cases.

Below we show that the boundary of the asymptotic stability domain of a circulatory system perturbed by small dissipative and gyroscopic forces as well as that of a gyroscopic system perturbed by weak dissipative and non-conservative positional forces possesses singularities such as “Dihedral angle” and “Whitney umbrella” governing stabilization and destabilization. In case of two degrees of freedom, we find approximations of the stability boundary near the singularities and obtain explicit estimates of the critical parameters in terms of the invariants of matrices of the system. This allows us to get a unified picture of the domain of asymptotic stability that clarifies a role of various forces in the destabilization paradox and helps with an establishment of connections with previously known results. As an example, we investigate the asymptotic stability domain of the modified Maxwell–Bloch equations with an application to the stability problems of gyroscopic systems with stationary and rotating damping.

2. A circulatory system with small velocity-dependent forces

We begin the study of stability of system (1) with imperfect reversible case ($\delta, \Omega \ll \nu$). In spite of the fact that the similar problem has been investigated recently by means of the perturbation theory of multiple eigenvalues for systems with arbitrary many degrees of freedom and for distributed systems [32,33,38–43], we restrict our subsequent considerations to $m=2$ degrees of freedom. This allows us to catch significant details remaining valid in the general case, and to solve the problem almost exactly, providing a reference necessary for the improvement of the approximation techniques based on the analysis of eigenvalues. System (1) with two degrees of freedom has also an independent interest because it contains actual low-dimensional models of dynamical systems such as disk brakes, space tethers, and spinning tops [20,22,23,30,31,36,37,44,46,47].

2.1. Stability of a circulatory system

Stability of system (1) is determined by its characteristic polynomial (9), which in case of two degrees of freedom has a convenient form provided by the Leverrier–Barnett algorithm [49]

$$P(\lambda, \delta, \nu, \Omega) = \lambda^4 + \delta \operatorname{tr} \mathbf{D}\lambda^3 + (\operatorname{tr} \mathbf{K} + \delta^2 \det \mathbf{D} + \Omega^2)\lambda^2 + (\delta(\operatorname{tr} \mathbf{K} \operatorname{tr} \mathbf{D} - \operatorname{tr} \mathbf{KD}) + 2\Omega\nu)\lambda + \det \mathbf{K} + \nu^2, \quad (10)$$

where without loss of generality we assume that the matrices **G** and **N** are equal to the 2×2 unit symplectic matrix.

In the absence of damping and gyroscopic forces ($\delta=\Omega=0$) system (1) is circulatory, and the characteristic polynomial (10) has four roots $-\lambda_+$, $-\lambda_-$, λ_-, and λ_+, where

$$\lambda_\pm = \sqrt{-\tfrac{1}{2}\operatorname{tr}\mathbf{K} \pm \tfrac{1}{2}\sqrt{(\operatorname{tr}\mathbf{K})^2 - 4(\det\mathbf{K} + \nu^2)}}. \quad (11)$$

Depending on the properties of the real symmetric matrix **K** and the magnitude ν of the non-conservative positional force, the eigenvalues (11) can be real, complex or purely imaginary implying instability or marginal stability in accordance with the following statement.

Proposition 1. *If* $\operatorname{tr}\mathbf{K} > 0$ *and* $\det\mathbf{K} \leqslant 0$, *circulatory system* (2) *with two degrees of freedom is stable for* $\nu_d^2 < \nu^2 < \nu_f^2$, *unstable by divergence for* $\nu^2 \leqslant \nu_d^2$, *and unstable by flutter for* $\nu^2 \geqslant \nu_f^2$, *where the critical values* ν_d *and* ν_f *are*

$$0 \leqslant \sqrt{-\det\mathbf{K}} =: \nu_d \leqslant \nu_f := \tfrac{1}{2}\sqrt{(\operatorname{tr}\mathbf{K})^2 - 4\det\mathbf{K}}. \quad (12)$$

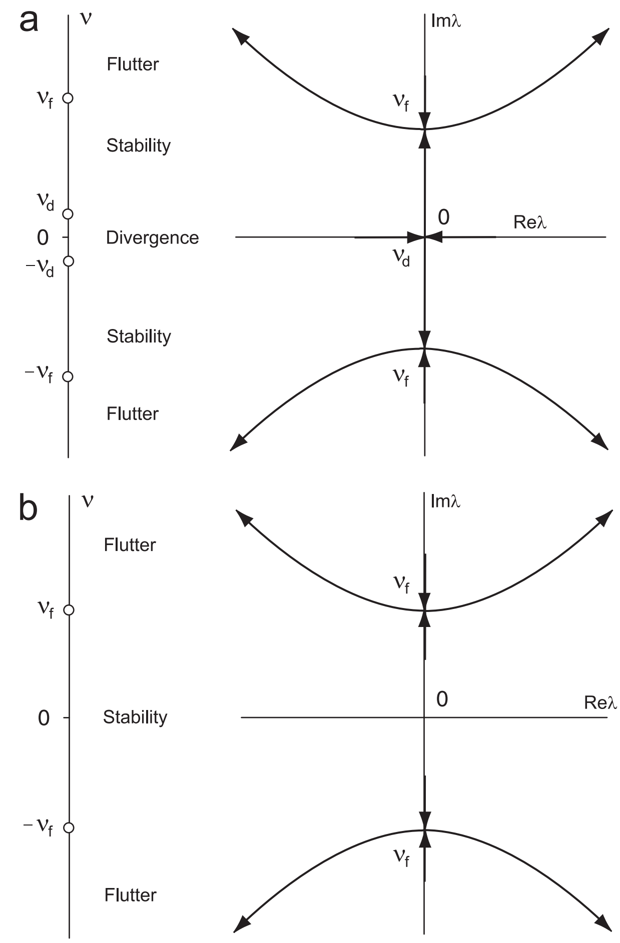

Fig. 1. Stability diagrams and trajectories of the eigenvalues for the increasing parameter $\nu > 0$ drawn for the circulatory system (2) with $\operatorname{tr}\mathbf{K} > 0$ and $\det\mathbf{K} < 0$ (a) and $\operatorname{tr}\mathbf{K} > 0$ and $\det\mathbf{K} > 0$ (b).

If $\operatorname{tr}\mathbf{K} > 0$ *and* $\det\mathbf{K} > 0$, *the circulatory system is stable for* $\nu^2 < \nu_f^2$ *and unstable by flutter for* $\nu^2 \geqslant \nu_f^2$.

If $\operatorname{tr}\mathbf{K} \leqslant 0$, *the system is unstable*.

The proof is a simple consequence of formula (11), reversible symmetry, and the fact that time dependence of solutions of Eq. (2) is given by $\exp(\lambda t)$ for simple eigenvalues λ, with an additional—polynomial in t—prefactor (secular terms) in case of multiple eigenvalues with the Jordan block. The solutions monotonously grow for positive real λ implying static instability (divergence), oscillate with an increasing amplitude for complex λ with positive real part (flutter), and remain bounded when λ is semi-simple and purely imaginary (stability). It is interesting to note that for a matrix **K** having two equal eigenvalues, the circulatory system (2) is unstable because $\nu_f = 0$, which is the statement of the Merkin theorem for circulatory systems with two degrees of freedom [10,47].

Stability diagrams and motion of eigenvalues in the complex plane for ν increasing from zero are presented in Fig. 1 . When $\operatorname{tr}\mathbf{K} > 0$ and $\det\mathbf{K} < 0$ there are two real and two purely imaginary eigenvalues at $\nu=0$, and the system is statically unstable, see Fig. 1(a). With the increase of ν both the imaginary and real eigenvalues are moving to the origin, until at $\nu=\nu_d$ the real pair merges and originates a double zero eigenvalue with the Jordan block. At $\nu=\nu_d$ the system is unstable because of the linear time dependence of a solution corresponding to $\lambda=0$. The further increase of ν yields splitting of the double zero eigenvalue into two purely imaginary eigenvalues. The imaginary eigenvalues of the same sign are then moving towards each other until at $\nu=\nu_f$ they originate a pair of double eigenvalues $\pm i\omega_f$ with the Jordan block, where

$$\omega_f = \sqrt{\tfrac{1}{2}\operatorname{tr}\mathbf{K}}. \tag{13}$$

At $\nu=\nu_f$ the system is unstable by flutter due to secular terms in its solutions. For $\nu>\nu_f$ the flutter instability is caused by two of the four complex eigenvalues lying on the branches of a hyperbolic curve

$$\operatorname{Im}\lambda^2 - \operatorname{Re}\lambda^2 = \omega_f^2. \tag{14}$$

The critical values ν_d and ν_f of the magnitude of the non-conservative positional force, constitute the boundaries between the divergence and stability domains and between the stability and flutter domains, respectively. For $\operatorname{tr}\mathbf{K}>0$ and $\det\mathbf{K}=0$ the divergence domain shrinks to a point $\nu_d=0$ and for $\operatorname{tr}\mathbf{K}>0$ and $\det\mathbf{K}>0$ there exist only stability and flutter domains as shown in Fig. 1(b). Obviously, the described picture is the same for negative ν with $\nu=-\nu_d$ and $\nu=-\nu_f$ indicating the boundaries of the divergence and flutter domains.

2.2. *The influence of small damping and gyroscopic forces on the stability of a circulatory system*

The one-dimensional domain of marginal stability of circulatory system (2) given by Proposition 1 is blowing up into a three-dimensional domain of asymptotic stability of system (1) in the space of the parameters δ, Ω, and ν. To find the domain of asymptotic stability we apply the criterion of Routh and Hurwitz in the form of Liénard and Chipart to the characteristic polynomial (10)

$$\delta\operatorname{tr}\mathbf{D}>0, \tag{15}$$

$$\operatorname{tr}\mathbf{K}+\delta^2\det\mathbf{D}+\Omega^2>0, \tag{16}$$

$$\det\mathbf{K}+\nu^2>0, \tag{17}$$

$$Q(\delta,\Omega,\nu)>0, \tag{18}$$

where

$$Q := \delta\operatorname{tr}\mathbf{D}(\operatorname{tr}\mathbf{K}+\delta^2\det\mathbf{D}+\Omega^2)(\delta(\operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D}-\operatorname{tr}\mathbf{KD})+2\Omega\nu) -(\delta\operatorname{tr}\mathbf{D})^2(\det\mathbf{K}+\nu^2)-(\delta(\operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D}-\operatorname{tr}\mathbf{KD}) +2\Omega\nu)^2. \tag{19}$$

Despite the explicit form, inequalities (15)–(18) do not possess an obvious interpretation. An alternative way is to use the qualitative theory of Arnold and then a perturbative approach utilizing smallness of parameters δ and Ω. For this purpose we remind that the stability problem for initial system (1) is equivalent to a stability problem for the first-order system, with the real $2m\times 2m$ matrix $\mathbf{A}$ defined by expression (4).

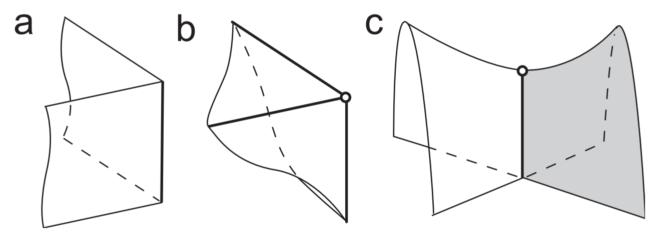

Fig. 2. Singularities *dihedral angle* (a), *trihedral angle* (b), and *deadlock of an edge* (or a half of the *Whitney umbrella* (c)) of the boundary of the asymptotic stability domain of a real three parameter matrix family.

As it has been established by Arnold [3], the boundary of the asymptotic stability domain of a multiparameter family of real matrices is not a smooth surface. Generically it possesses singularities corresponding to multiple eigenvalues with zero real part. In particular, for real matrices depending on three parameters, two different pairs of simple purely imaginary eigenvalues originate a singularity of the stability boundary, which is shaped as a *dihedral angle* in the parameter space, Fig. 2 (a). A double zero eigenvalue with the Jordan block and a pair of simple purely imaginary eigenvalues are responsible for the appearance of a *trihedral angle*, Fig. 2(b). A pair of double purely imaginary eigenvalues with the Jordan block corresponds to the singularity *deadlock of an edge*, which is a half of the *Whitney umbrella* surface [3], see Fig. 2(c).

Considering the asymptotic stability domain of system (1) in the space of the parameters δ, ν and Ω we know that the ν-axis is related to the unperturbed circulatory system (2). The parts of this axis belonging to the stability domain of system (2) and corresponding to two different pairs of simple purely imaginary eigenvalues, form edges of the dihedral angles on the surfaces that bound the asymptotic stability domain of system (1). At the points $\pm\nu_f$ of the ν-axis, corresponding to the stability–flutter boundary of system (2) there exists a pair of double purely imaginary eigenvalues with the Jordan block. Qualitatively, the asymptotic stability domain of system (1) in the space (δ,ν,Ω) near the ν-axis looks like a dihedral angle which becomes more acute while approaching the points $\pm\nu_f$. At these points the angle shrinks forming the deadlock of an edge. In case when the stability domain of the unperturbed circulatory system has a common boundary with the divergence domain, as shown in Fig. 1(a), the boundary of the asymptotic stability domain of the perturbed system (1) possesses the trihedral angle singularity at $\nu=\pm\nu_d$.

In the following, we improve the qualitative picture combining the direct analysis of the stability conditions (15)–(18) with a perturbation technique in the vicinity of the singularities located on the ν-axis. We find exact first-order approximations of the asymptotic stability domain, obtain an estimate of the critical value of the parameter ν as a function of δ and Ω, and show that this expression is reduced to a canonical equation of the surface with the Whitney umbrella singularity.

Note that the function $Q(\delta, v, \Omega)$ defined by Eq. (19) is a quadratic polynomial with respect to v. Solving the quadratic equation we write the stability condition (18) in the form

$$(v - v_{\rm cr}^-)(v - v_{\rm cr}^+) < 0, \tag{20}$$

where

$$v_{\rm cr}^{\pm}(\delta, \Omega) = \frac{\Omega b \pm \sqrt{\Omega^2 b^2 + ac}}{a}\,\delta, \tag{21}$$

and

$$a(\delta, \Omega) = 4\Omega^2 + \delta^2 (\operatorname{tr}\mathbf{D})^2,$$
$$b(\delta, \Omega) = \Omega^2 \operatorname{tr}\mathbf{D} - \operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D} + 2\operatorname{tr}\mathbf{KD} + \delta^2 \operatorname{tr}\mathbf{D}\det\mathbf{D},$$
$$c(\delta, \Omega) = (\operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D} - \operatorname{tr}\mathbf{KD})(\operatorname{tr}\mathbf{KD} + \delta^2 \operatorname{tr}\mathbf{D}\det\mathbf{D} + \Omega^2 \operatorname{tr}\mathbf{D}) - \det\mathbf{K}(\operatorname{tr}\mathbf{D})^2. \tag{22}$$

Analyzing conditions of asymptotic stability (15)–(17) and (20), we observe that the first two of them restrict the region of variation of parameters δ and Ω either to a half-plane $\delta \operatorname{tr}\mathbf{D} > 0$, if $\det\mathbf{D} \geqslant 0$, or to a space between the line $\delta = 0$ and one of the branches of a hyperbola $|\det\mathbf{D}|\,\delta^2 - \Omega^2 = 2\omega_{\rm f}^2$, if $\det\mathbf{D} < 0$. Provided that δ and Ω belong to the described domain, the asymptotic stability of system (1) is determined by inequalities (17) and (20), which impose limits on the variation of v.

As it follows from condition (17), for $\det\mathbf{K} \leqslant 0$ the divergence domain splits the domain of asymptotic stability into two non-intersecting parts, bounded by the planes $v = \pm v_{\rm d}$ and by the surfaces $v = v_{\rm cr}^{\pm}(\delta, \Omega)$. In comparison with the unperturbed system, the divergence boundary does not change because it is given by the same critical values $\pm v_{\rm d}$ which are independent from δ and Ω. For $\det\mathbf{K} > 0$ inequality (17) is fulfilled, and in accordance with condition (20) the asymptotic stability domain is contained between the surfaces $v = v_{\rm cr}^+(\delta, \Omega)$ and $v = v_{\rm cr}^-(\delta, \Omega)$.

The functions $v_{\rm cr}^{\pm}(\delta, \Omega)$ defined by expression (21) are singular at the origin due to vanishing denominator. Assuming $\Omega = \beta\delta$ and calculating a limit of these functions when δ tends to zero, we obtain

$$v_0^{\pm}(\beta) := \lim_{\delta \to 0} v_{\rm cr}^{\pm} = v_{\rm f}\frac{4\beta\beta_* \pm \operatorname{tr}\mathbf{D}\sqrt{(\operatorname{tr}\mathbf{D})^2 + 4(\beta^2 - \beta_*^2)}}{(\operatorname{tr}\mathbf{D})^2 + 4\beta^2}, \tag{23}$$

where

$$\beta_* := \frac{\operatorname{tr}(\mathbf{K} - \omega_{\rm f}^2 \mathbf{I})\mathbf{D}}{2v_{\rm f}}. \tag{24}$$

The limits $v_0^{\pm}(\beta)$ are real-valued functions of β if the radicand in expression (23) is non-negative.

Proposition 2. *Let $\lambda_1(\mathbf{D})$ and $\lambda_2(\mathbf{D})$ be eigenvalues of $\mathbf{D}$. Then,*

$$|\beta_*| \leqslant \frac{|\lambda_1(\mathbf{D}) - \lambda_2(\mathbf{D})|}{2}. \tag{25}$$

If $\mathbf{D}$ is semi-definite ($\det\mathbf{D} \geqslant 0$) or indefinite with

$$0 > \det\mathbf{D} \geqslant -\frac{(k_{12}(d_{22} - d_{11}) - d_{12}(k_{22} - k_{11}))^2}{4v_{\rm f}^2}, \tag{26}$$

then

$$|\beta_*| \leqslant \frac{|\operatorname{tr}\mathbf{D}|}{2}, \tag{27}$$

and the limits $v_0^{\pm}(\beta)$ are continuous real-valued functions of β. Otherwise, there exists an interval of discontinuity $\beta^2 < \beta_*^2 - (\operatorname{tr}\mathbf{D})^2/4$.

Proof. With the use of the definition of β_* (24), a series of transformations

$$\begin{aligned}\beta_*^2 - \frac{(\operatorname{tr}\mathbf{D})^2}{4} &= \frac{1}{4v_{\rm f}^2}\left(\frac{(k_{11} - k_{22})(d_{11} - d_{22})}{2} + 2k_{12}d_{12}\right)^2 \\ &\quad - \frac{(d_{11} + d_{22})^2}{4}\frac{((k_{11} - k_{22})^2 + 4k_{12}^2)}{4v_{\rm f}^2} \\ &= -\det\mathbf{D} - \frac{(k_{12}(d_{22} - d_{11}) - d_{12}(k_{22} - k_{11}))^2}{4v_{\rm f}^2}\end{aligned} \tag{28}$$

yields the expression

$$\beta_*^2 = \frac{(\lambda_1(\mathbf{D}) - \lambda_2(\mathbf{D}))^2}{4} - \frac{(k_{12}(d_{22} - d_{11}) - d_{12}(k_{22} - k_{11}))^2}{4v_{\rm f}^2}. \tag{29}$$

For real β_*, formula (29) implies inequality (25). The remaining part of the proposition follows from (28). □

To get an impression of the behavior of the functions $v_0^{\pm}(\beta)$, we calculate and plot them, normalized by $v_{\rm f}$, for the following positive-definite matrix $\mathbf{K}$ and indefinite matrix $\mathbf{D} = \mathbf{D}_i$, where $i = 1, 2, 3$:

$$\mathbf{K} = \begin{pmatrix} 27 & 3 \\ 3 & 5 \end{pmatrix}, \quad \mathbf{D}_1 = \begin{pmatrix} 6 & 3 \\ 3 & 1 \end{pmatrix},$$
$$\mathbf{D}_2 = \begin{pmatrix} 7 & \frac{4}{3}\sqrt{130} - 11 \\ \frac{4}{3}\sqrt{130} - 11 & 1 \end{pmatrix}, \quad \mathbf{D}_3 = \begin{pmatrix} 7 & 5 \\ 5 & 1 \end{pmatrix}. \tag{30}$$

The graphs of the functions $v_0^{\pm}(\beta)$ bifurcate with a change of $\det\mathbf{D}$. Indeed, since $\mathbf{D}_1$ satisfies the strict inequality (26), the limits are continuous functions with separated graphs, as shown in Fig. 3(a). Expression (26) is an equality for the matrix $\mathbf{D}_2$. Consequently, the functions $v_0^{\pm}(\beta)$ are continuous, with their graphs touching each other at the origin, Fig. 3(b). For the matrix $\mathbf{D}_3$, condition (26) is not fulfilled, and the functions are discontinuous. Their graphs, however, are joint together, forming continuous curves, see Fig. 3(c).

Except for a strongly pronounced bifurcation pattern, Fig. 3 shows that the calculated $v_0^{\pm}(\beta)$ are bounded functions of β, not exceeding the critical values $\pm v_{\rm f}$ of the unperturbed circulatory system.

Proposition 3.

$$|v_0^{\pm}(\beta)| \leqslant |v_0^{\pm}(\pm\beta_*)| = v_{\rm f}. \tag{31}$$

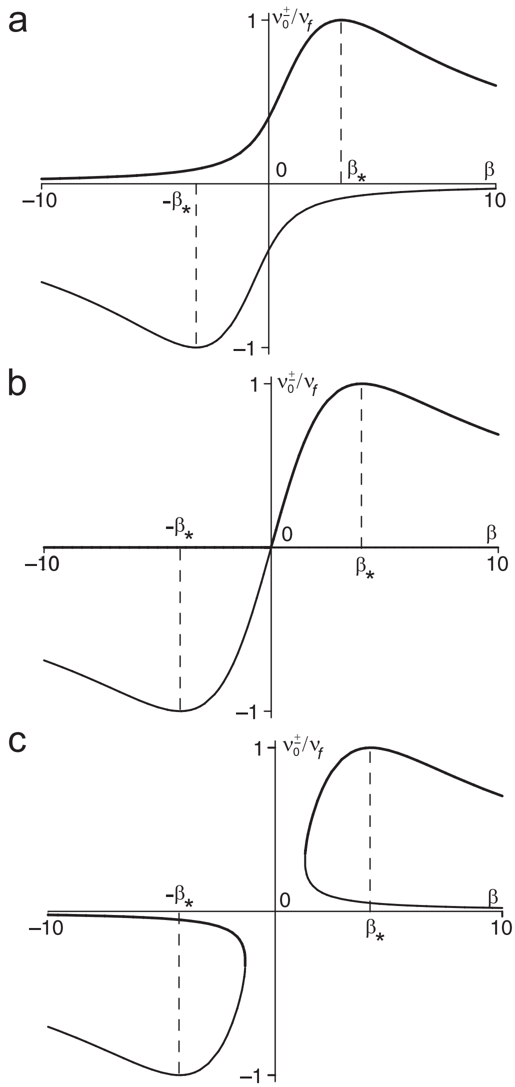

Fig. 3. The functions $\nu_0^+(\beta)$ (bold lines) and $\nu_0^-(\beta)$ (fine lines), and their bifurcation when $\mathbf{D}$ is changing.

Proof. Let us observe that $\mu_0^{\pm} := \nu_0^{\pm}/\nu_{\mathrm{f}}$ are roots of the quadratic equation

$$\nu_{\mathrm{f}}^2 a_\beta \mu^2 - 2\delta\Omega b_0 \nu_{\mathrm{f}} \mu - \delta^2 c_0 = 0, \tag{32}$$

where

$$\delta^2 a_\beta := a(\delta, \beta\delta) = \delta^2(4\beta^2 + (\mathrm{tr}\,\mathbf{D})^2),$$
$$b_0 := b(0,0) = 4\nu_{\mathrm{f}}\beta_*,$$
$$c_0 := c(0,0) = \nu_{\mathrm{f}}^2((\mathrm{tr}\,\mathbf{D})^2 - 4\beta_*^2). \tag{33}$$

According to the Schur criterion [50] all the roots μ of Eq. (32) are inside the closed unit disk, if

$$\delta^2 c_0 + \nu_{\mathrm{f}}^2 a_\beta = (\mathrm{tr}\,\mathbf{D})^2 + 4(\beta^2 - \beta_*^2) + (\mathrm{tr}\,\mathbf{D})^2 \geqslant 0,$$
$$2\delta\Omega\nu_{\mathrm{f}} b_0 + \nu_{\mathrm{f}}^2 a_\beta - \delta^2 c_0 = (\beta + \beta_*)^2 \geqslant 0,$$
$$-2\delta\Omega\nu_{\mathrm{f}} b_0 + \nu_{\mathrm{f}}^2 a_\beta - \delta^2 c_0 = (\beta - \beta_*)^2 \geqslant 0. \tag{34}$$

The first of conditions (34) is satisfied for real $\nu_0^{\pm}$, implying $|\mu_0^{\pm}(\beta)| \leqslant 1$ with $|\mu_0^+(\beta_*)| = |\mu_0^-(-\beta_*)| = 1$. □

Therefore, the magnitude of the non-conservative positional force at the onset of flutter for system (1) with vanishingly small dissipative and gyroscopic forces, does not exceed that of the circulatory system (2), demonstrating a jump in the critical load which is characteristic of the destabilization paradox in Ziegler's form.

Another characteristic feature of this destabilization paradox is an abrupt change of the critical frequency of the onset of flutter due to an infinitesimally small damping [8,13]. This phenomenon is especially important in the problems of acoustics of friction, such as disk brake squeal suppression [30,31,43,46]. An explicit and exact expression for the critical frequency follows directly from the characteristic polynomial (10)

$$\omega_{\mathrm{cr}}(\delta, \Omega, \nu) = \pm\omega_{\mathrm{f}}\sqrt{1 + 2\frac{\nu\beta - \nu_{\mathrm{f}}\beta_*}{\omega_{\mathrm{f}}^2\,\mathrm{tr}\,\mathbf{D}}}$$
$$= \pm\omega_{\mathrm{f}} \pm \frac{\nu_{\mathrm{f}}(\beta - \beta_*) + \beta_*(\nu - \nu_{\mathrm{f}})}{\mathrm{tr}\,\mathbf{D}}$$
$$+ o(\beta - \beta_*, \nu - \nu_{\mathrm{f}}), \tag{35}$$

being in agreement with the results of the works [13,41,42].

It is remarkable that the limits $\nu_0^{\pm}(\beta)$ of the critical values of the circulatory parameter $\nu_{\mathrm{cr}}^{\pm}(\delta, \Omega)$, which are complicated functions of δ and Ω, depend only on the ratio $\beta = \Omega/\delta$, defining the direction of approaching zero in the plane (δ, Ω). Along the directions $\beta = \beta_*$ and $\beta = -\beta_*$, the limits coincide with the critical flutter loads of the unperturbed circulatory system (2) in such a way that $\nu_0^+(\beta_*) = \nu_{\mathrm{f}}$ and $\nu_0^-(-\beta_*) = -\nu_{\mathrm{f}}$. Power series expansions of the functions $\nu_0^{\pm}(\beta)$ around $\beta = \pm\beta_*$ (with the radius of convergence not exceeding $|\mathrm{tr}\,\mathbf{D}|/2$) give simple estimates of the jumps in the critical load

$$\nu_{\mathrm{f}} - \nu_0^+(\beta) = \nu_{\mathrm{f}}\frac{2}{(\mathrm{tr}\,\mathbf{D})^2}(\beta - \beta_*)^2 + o((\beta - \beta_*)^2),$$
$$\nu_{\mathrm{f}} + \nu_0^-(\beta) = \nu_{\mathrm{f}}\frac{2}{(\mathrm{tr}\,\mathbf{D})^2}(\beta + \beta_*)^2 + o((\beta + \beta_*)^2). \tag{36}$$

If we leave only second-order terms in expressions (36) and then substitute $\beta = \Omega/\delta$, we get equations of the form $Z = X^2/Y^2$, which is canonical for the singular surface known as the Whitney umbrella [3]. These equations give simple approximations of the boundary of the asymptotic stability domain of system (1) in the vicinity of the points $(0, 0, \pm\nu_{\mathrm{f}})$ in the space of the parameters (δ, Ω, ν), and confirm the qualitative picture following from the Arnold theory [3].

A more general way of investigation of the stability boundary near singularities is based on the perturbation theory for eigenvalues [8,13–18,20,22,25,28,32,33,38–43]. For a fixed ν a simple root $\lambda = \mathrm{i}\omega(\nu)$ of the characteristic polynomial $P(\lambda, \delta, \Omega, \nu)$, calculated at $\delta = \Omega = 0$, is expanded into the Taylor series

$$\lambda(\delta, \Omega, \nu) - \mathrm{i}\omega(\nu) = -\delta\frac{\partial_\delta P(\nu)}{\partial_\lambda P(\nu)} - \Omega\frac{\partial_\Omega P(\nu)}{\partial_\lambda P(\nu)} + o(\delta, \Omega). \tag{37}$$

It is evident, that equation $\mathrm{Re}(\lambda(\delta, \Omega, \nu)) = 0$ defines a curve in the plane (δ, Ω), which contributes to the boundary of the asymptotic stability domain, corresponding to some constant value of the parameter ν. A linear approximation to this curve in the vicinity of the origin is given by the expression

$$\mathrm{Re}(\delta\partial_\delta P(\nu) + \Omega\partial_\Omega P(\nu)) = 0, \quad (38)$$

which for the polynomial (10) reads as

$$\delta(2\nu_f\beta_* + (\omega^2(\nu) - \omega_f^2)\,\mathrm{tr}\,\mathbf{D}) - 2\Omega\nu = 0. \quad (39)$$

Eqs. (38), (39) give correct linear approximation of the boundary of the asymptotic stability domain, if the functions $\omega(\nu)$ are known exactly, as it is in our case, where Eqs. (11)–(13) yield

$$\omega^2(\nu) = \omega_f^2 \pm \sqrt{\nu_f^2 - \nu^2}. \quad (40)$$

Substituting (40) into (39), we obtain

$$\Omega = \frac{\nu_f}{\nu}\left[\beta_* \pm \frac{\mathrm{tr}\,\mathbf{D}}{2}\sqrt{1 - \frac{\nu^2}{\nu_f^2}}\right]\delta, \quad (41)$$

which is simply formula (23) inverted with respect to $\beta = \Omega/\delta$.

In general, for systems with $m \geqslant 2$ degrees of freedom, explicit analytical expressions like (40) cannot be found. Instead, numerical data can be used [13], or, alternatively, power series expansions of $\omega(\nu)$ in the vicinity of $\nu = \nu_f$, which in case of a double eigenvalue are given by the Newton–Puiseux series [28,32]

$$\omega(\nu) = \omega_f \pm \sqrt{2\frac{\partial_\nu P}{\partial^2_{\lambda\lambda} P}(\nu - \nu_f)} + O((\nu - \nu_f)). \quad (42)$$

As a payment for the generality of the approach, Eq. (39) and expression (42), evaluated for polynomial (10), give only approximations (36) of formula (23). Such approximations for general finite-dimensional and distributed imperfect reversible systems were obtained in [32,38,40–43].

Being based on the linear approximation (41), we study an asymptotic behavior of the stability domain in the vicinity of the origin in the plane (δ, Ω) for various ν. This give us better understanding of the shape of the stability domain in the space of the parameters δ, Ω, and ν.

For our purpose it is enough to consider only the case when $\mathrm{tr}\,\mathbf{K} > 0$ and $\det\mathbf{K} > 0$, so that $-\nu_f < \nu < \nu_f$, because for $\det\mathbf{K} \leqslant 0$ the region $\nu^2 < \nu_d^2 \leqslant \nu_f^2$ is unstable and should be excluded.

For $\nu^2 < \nu_f^2$, the radicand in expression (41) is real and nonzero, so that in the first approximation, the domain of asymptotic stability is contained in the angle between two lines intersecting at the origin, as depicted in Fig. 4 (central column). When ν approaches the critical values $\pm\nu_f$, the angle becomes more acute until at $\nu = \nu_f$ or $\nu = -\nu_f$ it degenerates to a single line $\Omega = \delta\beta_*$ or $\Omega = -\delta\beta_*$, respectively. For $\beta_* \neq 0$ these lines are not parallel to each other, and due to inequality (25) they never stay vertical, see Fig. 4 (right column). However, the degeneration can be lifted already in the second approximation

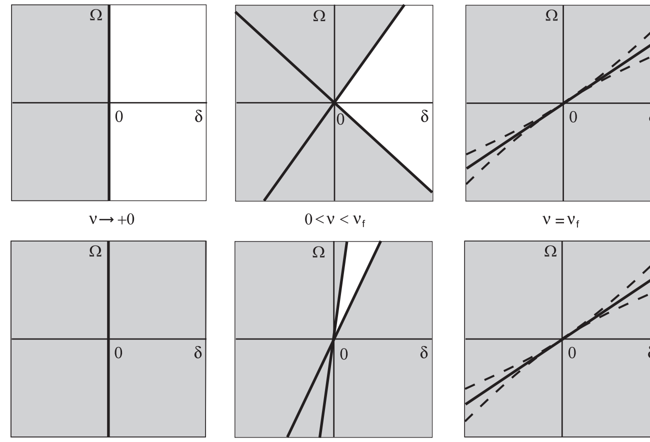

Fig. 4. For various ν, bold lines show linear approximations to the boundary of the asymptotic stability domain (white) of system (1) in the vicinity of the origin in the plane (δ, Ω), when $\mathrm{tr}\,\mathbf{K} > 0$ and $\det\mathbf{K} > 0$, and condition (26) is fulfilled (upper row) or fails (lower row).

$$\Omega = \pm\delta\beta_* \pm \frac{\omega_f\,\mathrm{tr}\,\mathbf{D}\sqrt{\det\mathbf{D} + \beta_*^2}}{2\nu_f}\delta^2 + O(\delta^3). \quad (43)$$

If the radicand is positive, Eq. (43) defines two curves touching each other at the origin, in such a way that the domain of asymptotic stability has a cuspidal form, as shown in Fig. 4 by dashed lines. Inside the cusp the critical value of the circulatory parameter ν of the imperfect reversible system (1) is not less than that of the circulatory system (2). One can conclude, that an accurate choice of the small velocity-dependent forces can enlarge the stability range for ν.

The evolution of the domain of asymptotic stability, when ν goes from $\pm\nu_f$ to zero, depends on the properties of the matrix $\mathbf{D}$ and is governed by inequality (26). In case when the inequality is fulfilled, we have $|2\beta_*| \leqslant \mathrm{tr}\,\mathbf{D}$, and the angle between two lines (41) is getting wider, tending to π for $\nu \to 0$, as shown in Fig. 4 (upper left). Otherwise, the angle reaches a maximum for some $\nu^2 < \nu_f^2$ and then shrinks to a single line $\delta = 0$ at $\nu = 0$, Fig. 4 (lower left). We note that these degenerations differ from that occurred when $\nu \to \pm\nu_f$, because they do not depend on the order of approximation. Indeed, at $\nu = 0$ the Ω-axis corresponds to a marginally stable gyroscopic system, and thus it forms an edge of the dihedral angle singularity of the boundary of the domain of asymptotic stability.

Since the linear approximation to the asymptotic stability domain does not contain the Ω-axis at any $\nu \neq 0$, small gyroscopic forces cannot stabilize a circulatory system in the absence of damping forces ($\delta = 0$), which is in agreement with theorems of Lakhadanov and Karapetyan [11,12]. Another interesting interpretation of Fig. 4 is that a conservative system with $\mathbf{K} > 0$ can be made asymptotically stable by small gyroscopic and damping forces with semi-definite or indefinite matrix $\mathbf{D}$, satisfying condition (26). For indefinite matrices $\mathbf{D}$ violating inequality (26), the asymptotic stability can be reached only in the presence of gyroscopic, damping, and circulatory

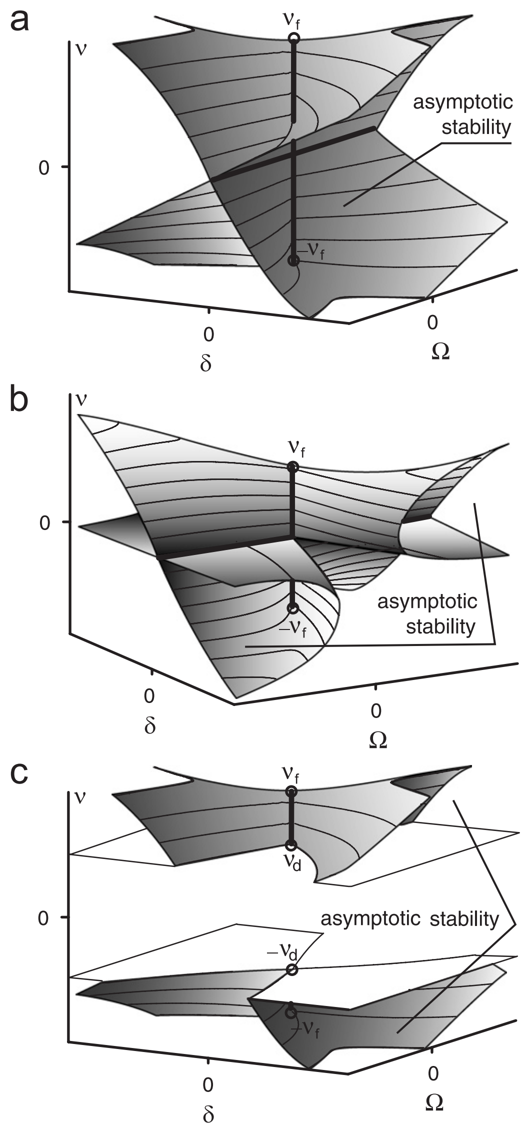

Fig. 5. Blowing the stability domain of circulatory system (2) up to the domain of asymptotic stability of system (1) with the singularities *Whitney umbrella*, *dihedral angle*, and *trihedral angle* when $\operatorname{tr}\mathbf{K}>0$, $\det\mathbf{K}>0$ and condition (26) is fulfilled (a) or fails (b), and when $\operatorname{tr}\mathbf{K}>0$ and $\det\mathbf{K}<0$ (c).

forces. Therefore, a stable circulatory system with two degrees of freedom can be made asymptotically stable by the proper combination of gyroscopic and damping forces with real symmetric matrix **D** having *arbitrary* structure, as it is clearly seen from Eq. (41) and Fig. 4. Therefore, at least in two dimensions the requirement of definiteness of the matrix **D** established in [35] is not necessary for the stabilization of circulatory systems by gyroscopic and damping forces.

Combining our analytical results with the qualitative picture based on the singularity theory, we conclude that there are three typical configurations of the surfaces $v=v_{\rm cr}^{\pm}(\delta,\Omega)$ and thus three typical configurations of the asymptotic stability domain of system (1) in the vicinity of the v-axis. The parts of the surfaces $v=v_{\rm cr}^{\pm}(\delta,\Omega)$ bounding the domain of asymptotic stability, belong to the half-space $\delta\operatorname{tr}\mathbf{D}>0$, as depicted in Fig. 5.

The first configuration corresponds to a positive-definite matrix **K** and to a semi-definite or indefinite matrix **D**, which satisfies condition (26). We see that addition of small damping and gyroscopic forces blows the stability interval of a circulatory system $v^2<v_{\rm f}^2$ up to a three-dimensional region bounded by a surface with singularities, Fig. 5(a). The stability interval of a circulatory system itself becomes an edge of a dihedral angle, formed due to intersection of two smooth surfaces. At $v=0$ the angle of the intersection reaches its maximum (π), creating another edge along the Ω-axis. While approaching the points $\pm v_{\rm f}$, the angle becomes more acute and ends up with the deadlock of an edge, which is a part of the Whitney umbrella singularity, Fig. 5(a).

Changing the matrix **D** in such a way that it approaches the threshold of condition (26), we deform the asymptotic stability domain. Two smooth parts of the stability boundary corresponding to negative and positive v are going towards each other until they touch, when **D** is at the threshold. After **D** violates condition (26) this temporary glued configuration collapses into two pockets of asymptotic stability as shown in Fig. 5(b). Each of the two pockets has a deadlock of an edge as well as two edges which meet at the origin. This complicated geometry is responsible for the difficulties of stabilization by damping forces with indefinite matrix **D**. Moreover, according to inequality (19) in this case the magnitude of the damping forces is limited from above. For example, if $v=\Omega=0$, then $\delta^2<\delta_{\rm cr}^2$ where

$$\delta_{\rm cr}=\sqrt{-\frac{\operatorname{tr}(\mathbf{KD}-\lambda_1(\mathbf{K})\mathbf{D})\operatorname{tr}(\mathbf{KD}-\lambda_2(\mathbf{K})\mathbf{D})}{\operatorname{tr}\mathbf{D}\det\mathbf{D}\,(\operatorname{tr}\mathbf{KD}-\operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D})}},\tag{44}$$

and $\lambda_1(\mathbf{K})$ and $\lambda_2(\mathbf{K})$ are eigenvalues of the matrix **K**. We note that the formula (44) generalizes the result of Freitas et al. [27], which was obtained for a diagonal matrix of potential forces.

The third configuration corresponds to an indefinite matrix **K** with $\operatorname{tr}\mathbf{K}>0$ and $\det\mathbf{K}<0$, Fig. 5(c). In this case the condition $v^2>v_{\rm d}^2$ divides the domain of asymptotic stability into two parts, corresponding to positive and negative v. The intervals of v-axis form edges of dihedral angles which end up with the deadlocks at $v=\pm v_{\rm f}$ and with the trihedral angles at $v=\pm v_{\rm d}$, Fig. 5(c). Qualitatively, this configuration does not depend on the properties of the matrix **D**.

We see that the asymptotic stability domain of the nonconservative system (1) naturally incorporates not only the interval of marginal stability of circulatory system (2) but also that of gyroscopic system (3) in such a manner that they serve as singularities of its boundary. The imperfect reversible system is therefore intimately related with the imperfect Hamiltonian one, whose domain of asymptotic stability we study in the next section.

3. A gyroscopic system with weak damping and circulatory forces

In 1879, Thomson and Tait showed that a statically unstable potential system, which has been stabilized by gyroscopic forces could be destabilized by the introduction of small damping [4]. Later it has been observed that many statically unstable gyropendulums enjoy robust stability at high speeds [22]. Since the idea that damping is completely absent in a real system could not be accepted, it has been understood that the nature of damping itself may be different from that assumed in the stationary damping model. In 1933, Smith introduced a concept of rotating damping and showed on a planar model of a flexible rotor, that the critical angular velocity depends on the ratio of the coefficients of stationary and rotating damping [6,22]. Contrary to the stationary damping, which is a velocity-dependent force, the rotating one is also proportional to the displacements by a non-conservative way and thus contributes not only to the matrix $\mathbf{D}$ in Eq. (1), but to the matrix $\mathbf{N}$ as well. This leads to a problem of perturbation of conservative gyroscopic system (3) by weak dissipative and non-conservative positional forces, attracted recently new attention [19,22,23,35–37,44,46–48].

3.1. Stability of a conservative gyroscopic system

First we consider stability of gyroscopic system (3). In the absence of dissipative and circulatory forces ($\delta = \nu = 0$), the characteristic polynomial (10) has four roots $\pm\lambda_\pm$, where

$$\lambda_\pm = \sqrt{-\tfrac{1}{2}(\operatorname{tr}\mathbf{K} + \Omega^2) \pm \tfrac{1}{2}\sqrt{(\operatorname{tr}\mathbf{K} + \Omega^2)^2 - 4\det\mathbf{K}}}. \tag{45}$$

Analysis of these eigenvalues yields the following result, see e.g. [24].

Proposition 4. *If* $\det\mathbf{K} > 0$ *and* $\operatorname{tr}\mathbf{K} < 0$, *gyroscopic system* (3) *with two degrees of freedom is unstable by divergence for* $\Omega^2 < \Omega_0^{-2}$, *unstable by flutter for* $\Omega_0^{-2} \leqslant \Omega^2 \leqslant \Omega_0^{+2}$, *and stable for* $\Omega_0^{+2} < \Omega^2$, *where the critical values* Ω_0^- *and* Ω_0^+ *are*

$$0 \leqslant \sqrt{-\operatorname{tr}\mathbf{K} - 2\sqrt{\det\mathbf{K}}} =: \Omega_0^- \leqslant \Omega_0^+ := \sqrt{-\operatorname{tr}\mathbf{K} + 2\sqrt{\det\mathbf{K}}}. \tag{46}$$

If $\det\mathbf{K} > 0$ *and* $\operatorname{tr}\mathbf{K} > 0$, *the gyroscopic system is stable for any* Ω.

If $\det\mathbf{K} \leqslant 0$, *the system is unstable*.

We note that the last two statements are simply the theorems of Routh [5] and Thomson and Tait [4], respectively. The remaining part of the proposition follows from Eq. (45), represented in the form

$$\lambda_\pm = \sqrt{-\tfrac{1}{2}(\Omega^2 - \tfrac{1}{2}(\Omega_0^{-2} + \Omega_0^{+2})) \pm \tfrac{1}{2}\sqrt{(\Omega^2 - \Omega_0^{-2})(\Omega^2 - \Omega_0^{+2})}}. \tag{47}$$

Indeed, at $\Omega = 0$ there are in general four real roots $\pm\lambda_\pm = \pm(\Omega_0^+ \pm \Omega_0^-)/2$ and system (3) is statically unstable. With the increase of Ω^2 the distance $\lambda_+ - \lambda_-$ between the two roots of the same sign is getting smaller. The roots are moving towards each other until they merge at $\Omega^2 = \Omega_0^{-2}$ with the origination of a pair of double real eigenvalues $\pm\omega_0$ with the Jordan blocks, where

$$\omega_0 = \tfrac{1}{2}\sqrt{\Omega_0^{+2} - \Omega_0^{-2}} = \sqrt[4]{\det\mathbf{K}} > 0. \tag{48}$$

Further increase of Ω^2 yields splitting of $\pm\omega_0$ to two couples of complex-conjugate eigenvalues lying on the circle

$$\operatorname{Re}\lambda^2 + \operatorname{Im}\lambda^2 = \omega_0^2. \tag{49}$$

The complex eigenvalues move along the circle until at $\Omega^2 = \Omega_0^{+2}$ they reach the imaginary axis and originate a complex-conjugate pair of double purely imaginary eigenvalues $\pm\mathrm{i}\omega_0$. For $\Omega^2 > \Omega_0^{+2}$ the double eigenvalues split into four simple purely imaginary eigenvalues which do not leave the imaginary axis, Fig. 6.

Thus, the system (3) with $\mathbf{K} < 0$ is statically unstable for $\Omega \in (-\Omega_0^-, \Omega_0^-)$, it is dynamically unstable for $\Omega \in [-\Omega_0^+, -\Omega_0^-] \cup [\Omega_0^-, \Omega_0^+]$, and it is stable (gyroscopic stabilization) for $\Omega \in (-\infty, -\Omega_0^+) \cup (\Omega_0^+, \infty)$, see Fig. 6. The values of the gyroscopic parameter $\pm\Omega_0^-$ define the boundary between the divergence and flutter domains while the values $\pm\Omega_0^+$ originate the flutter–stability boundary.

3.2. The influence of small damping and non-conservative positional forces on the stability of a gyroscopic system

Let us establish a relationship between the one-dimensional stability domain of gyroscopic system (3) given by Proposition 4, and the domain of asymptotic stability of imperfect Hamiltonian system (1) defined in the space of the parameters δ, ν, and Ω by conditions (15)–(17) and (20).

Observing that inequality (17) is fulfilled for $\det\mathbf{K} > 0$ and inequality (15) simply restricts the region of variation of δ to the half-plane $\delta\operatorname{tr}\mathbf{D} > 0$, we focus our analysis on conditions (16) and (20).

Let us consider the asymptotic stability domain in the plane of the parameters δ and ν in the vicinity of the origin, assuming that $\Omega \neq 0$ is fixed. Taking into account the structure of coefficients (22) and leaving the linear terms with respect to δ in the Taylor expansions of the functions $\nu_{cr}^\pm(\delta, \Omega)$, we get the equations determining a linear approximation to the stability boundary

$$\begin{aligned}\nu &= \frac{\operatorname{tr}\mathbf{KD} - \operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D} - \operatorname{tr}\mathbf{D}\lambda_\pm^2(\Omega)}{2\Omega}\delta \\ &= \frac{2\operatorname{tr}\mathbf{KD} + \operatorname{tr}\mathbf{D}(\Omega^2 - \operatorname{tr}\mathbf{K}) \pm \operatorname{tr}\mathbf{D}\sqrt{(\Omega^2 + \operatorname{tr}\mathbf{K})^2 - 4\det\mathbf{K}}}{4\Omega}\delta,\end{aligned} \tag{50}$$

where the eigenvalues $\lambda_\pm(\Omega)$ are given by formula (45).

For $\det\mathbf{K} > 0$ and $\operatorname{tr}\mathbf{K} > 0$ the gyroscopic system is stable at any Ω. Consequently, the coefficients $\lambda_\pm^2(\Omega)$ are always real, and Eqs. (50) define in general two lines intersecting at the origin, Fig. 7. Since $\operatorname{tr}\mathbf{K} > 0$, inequality (16) is satisfied for $\det\mathbf{D} \geqslant 0$, and it gives an upper bound of δ^2 for $\det\mathbf{D} < 0$. Thus, according to conditions (15) and (20), a linear approximation to the domain of asymptotic stability near the origin in the plane (δ, ν), is an angle between two lines (50), as shown in Fig. 7. With the change of Ω the size of the angle is varying

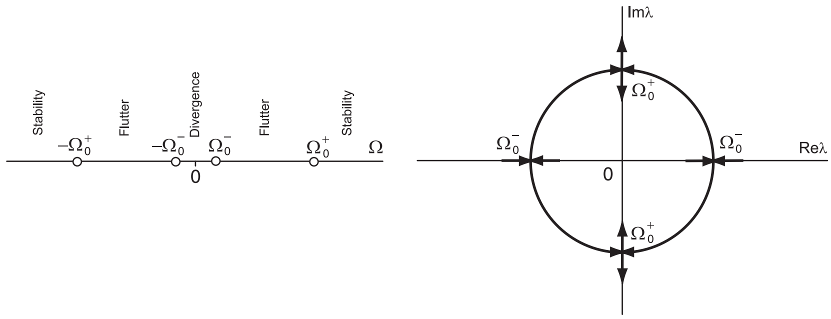

Fig. 6. Stability diagram for the conservative gyroscopic system with $\mathrm{tr}\,\mathbf{K}<0$ and $\det\mathbf{K}>0$ (left) and the corresponding trajectories of the eigenvalues in the complex plane for the increasing parameter $\Omega>0$ (right).

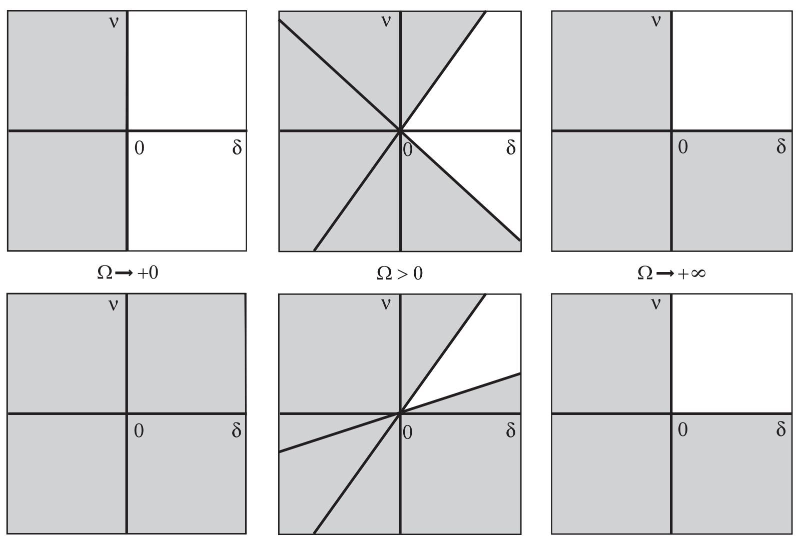

Fig. 7. For various Ω, bold lines show linear approximations to the boundary of the asymptotic stability domain (white) of system (1) in the vicinity of the origin in the plane (δ, ν), when $\mathrm{tr}\,\mathbf{K}>0$ and $\det\mathbf{K}>0$, and condition (26) is fulfilled (upper row) or fails (lower row).

and moreover, the stability domain rotates as a whole about the origin.

When $\Omega\to\infty$, the size of the angle defined by Eqs. (20) and (50) tends to $\pi/2$ in such a way that the stability domain fits one of the four quadrants of the parameter plane, as shown in Fig. 7 (right column). To study the shape of the stability domain at $\Omega\to 0$ we note that

$$|2\,\mathrm{tr}\,\mathbf{KD}-\mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{D}|-|\mathrm{tr}\,\mathbf{D}|\sqrt{(\mathrm{tr}\,\mathbf{K})^2-4\det\mathbf{K}}\leqslant 0, \tag{51}$$

if $\mathbf{D}$ satisfies condition (26). Consequently, the angle between the lines (50) tends to π, Fig. 7 (upper left). In this case the domain of asymptotic stability spreads over two quadrants and contains the δ-axis. Otherwise, the left side of inequality (51) is positive and at $\Omega\to 0$ the angle tends to zero, Fig. 7 (lower left). In these conditions the stability domain always belongs to one quadrant and does not contain δ-axis. The latter means that in the absence of non-conservative positional forces, gyroscopic system (3) with $\mathbf{K}>0$ cannot be made asymptotically stable by damping forces with indefinite matrix $\mathbf{D}$ violating inequality (26), which is also visible in the three-dimensional picture of Fig. 5(b).

The domain of asymptotic stability of imperfect Hamiltonian system (1) with $\mathbf{K}>0$ and $\mathbf{D}$ satisfying inequality (26), in the space of the three parameters δ, ν, and Ω has the form of a dihedral angle with the Ω-axis as its edge, as shown in Fig. 5(a). With the increase in $|\Omega|$, the section of the domain by the plane $\Omega=\mathrm{const}$ is getting more narrow and is rotating about the origin so that the points of the parameter plane (δ, ν) that where stable at lower $|\Omega|$ can lose their stability for the higher absolute values of the gyroscopic parameter. This geometry of the stability domain describes the mechanism of *gyroscopic destabilization* of a statically stable conservative system in the presence of damping and non-conservative positional forces.

To study the case when $\mathbf{K}<0$ we use expressions (46)–(48), and write Eqs. (50) in the form

$$\nu=\frac{\Omega_0^+}{\Omega}\left[\gamma_*+\frac{\mathrm{tr}\,\mathbf{D}}{4}\sqrt{\frac{\Omega^2}{\Omega_0^{+2}}-1}\times\left(\sqrt{\Omega^2-\Omega_0^{+2}}\pm\sqrt{\Omega^2-\Omega_0^{-2}}\right)\right]\delta, \tag{52}$$

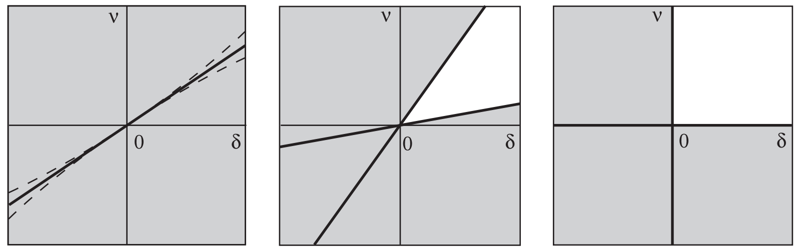

Fig. 8. For various Ω, bold lines show linear approximations to the boundary of the asymptotic stability domain (white) of system (1) in the vicinity of the origin in the plane (δ, ν), when $\mathrm{tr}\,\mathbf{K}<0$ and $\det\mathbf{K}>0$.

where

$$\gamma_* := \frac{\mathrm{tr}\,[\mathbf{K}+({\Omega_0^+}^2-\omega_0^2)\mathbf{I}]\mathbf{D}}{2\Omega_0^+}. \tag{53}$$

Proposition 5. *Let $\lambda_1(\mathbf{D})$ and $\lambda_2(\mathbf{D})$ be eigenvalues of $\mathbf{D}$. Then,*

$$|\gamma_*| \leqslant \Omega_0^+\frac{|\lambda_1(\mathbf{D})+\lambda_2(\mathbf{D})|}{4} + \Omega_0^-\frac{|\lambda_1(\mathbf{D})-\lambda_2(\mathbf{D})|}{4}. \tag{54}$$

Proof. Indeed, using the Cauchy–Schwarz inequality, after a series of transformations we obtain

$$|\gamma_*| \leqslant \Omega_0^+\frac{|\mathrm{tr}\,\mathbf{D}|}{4} + \frac{\mathrm{tr}(\mathbf{K}-(\mathrm{tr}\,\mathbf{K}/2)\mathbf{I})(\mathbf{D}-(\mathrm{tr}\,\mathbf{D}/2)\mathbf{I})}{2\Omega_0^+}$$
$$\leqslant \Omega_0^+\frac{|\mathrm{tr}\,\mathbf{D}|}{4} + \frac{|\lambda_1(\mathbf{K})-\lambda_2(\mathbf{K})||\lambda_1(\mathbf{D})-\lambda_2(\mathbf{D})|}{4\Omega_0^+}. \tag{55}$$

Taking into account that $|\lambda_1(\mathbf{K})-\lambda_2(\mathbf{K})| = \Omega_0^-\Omega_0^+$, we get inequality (54). □

Analyzing expression (52) we find that it is real-valued when $\Omega^2 \geqslant {\Omega_0^+}^2$ or $\Omega^2 \leqslant {\Omega_0^-}^2$. For sufficiently small $|\delta|$ the first inequality implies stability condition (16), whereas the last inequality contradicts it. Consequently, the domain of asymptotic stability is determined by the inequalities (15) and (20), and its linear approximation in the vicinity of the origin in the (δ, ν)-plane has the form of an angle with the boundaries given by Eq. (52). For Ω tending to infinity the angle expands to $\pi/2$, whereas for $\Omega=\Omega_0^+$ or $\Omega=-\Omega_0^+$ it degenerates to a single line $\nu=\delta\gamma_*$ or $\nu=-\delta\gamma_*$, respectively. For $\gamma_* \neq 0$ these lines are not parallel to each other, and due to inequality (55) they never stay vertical, see Fig. 8 (left). This degeneration can, however, be removed already in the second-order approximation

$$\nu = \pm\delta\gamma_* \pm \frac{\mathrm{tr}\,\mathbf{D}\sqrt{\omega_0^2\det\mathbf{D}-\gamma_*^2}}{2\Omega_0^+}\delta^2 + O(\delta^3). \tag{56}$$

If the radicand in Eq. (56) is positive, they describe two smooth curves touching each other at the origin in such a way that the stability domain has a form of a cusp, shown by dashed lines in Fig. 8 (left).

Therefore, gyroscopic stabilization of statically unstable conservative system with $\mathbf{K}<0$ can be improved up to asymptotic stability by small damping and circulatory forces, if their magnitudes are in the narrow region with the boundaries depending on Ω. The lower desirable absolute value of the critical gyroscopic parameter $\Omega_{\mathrm{cr}}(\delta, \nu)$ the poorer choice of the appropriate combinations of damping and circulatory forces.

The new critical value of the gyroscopic parameter $\Omega_{\mathrm{cr}}(\delta, \nu)$ can deviate significantly from that of the conservative gyroscopic system. To estimate it, we consider formula (52) in the vicinity of the points $(0, 0, \pm\Omega_0^+,)$ in the parameter space. Leaving only the terms, which are constant or proportional to $\sqrt{\Omega\pm\Omega_0^+}$ in both the numerator and denominator and assuming $\nu=\gamma\delta$, we write the equations separately for positive and negative Ω,

$$\Omega_{\mathrm{cr}}^+(\gamma) = \Omega_0^+ + \Omega_0^+\frac{2}{(\omega_0\,\mathrm{tr}\,\mathbf{D})^2}(\gamma-\gamma_*)^2 + o((\gamma-\gamma_*)^2), \tag{57}$$

$$\Omega_{\mathrm{cr}}^-(\gamma) = -\Omega_0^+ - \Omega_0^+\frac{2}{(\omega_0\,\mathrm{tr}\,\mathbf{D})^2}(\gamma+\gamma_*)^2 + o((\gamma+\gamma_*)^2). \tag{58}$$

It is remarkable that Eqs. (57) and (58) have the form $Z=X^2/Y^2$, canonical for the Whitney umbrella.

The domain of asymptotic stability of an imperfect Hamiltonian system (1) with $\mathbf{K}<0$ is shown in Fig. 9(a). The parts of the Ω-axis, which correspond to the stability domain of the unperturbed gyroscopic system, form an edge of the dihedral angle singularity of the stability boundary. The angle becomes more acute near the points $\pm\Omega_0^+$, at which it degenerates with the origination of the deadlock of an edge singularity. Qualitatively, the domain of asymptotic stability given by inequalities (15) and (20) consists of two pockets of two Whitney umbrellas, selected by the condition $\delta\,\mathrm{tr}\,\mathbf{D}>0$. Eq. (52) is a linear approximation to the stability boundary in the vicinity of the Ω-axis. Moreover, it describes in an implicit form a limit of the critical gyroscopic parameter $\Omega_{\mathrm{cr}}(\delta, \gamma\delta)$ when δ tends to zero, as a function of the ratio $\gamma=\nu/\delta$, Fig. 9(b).

As it follows from expressions (52), (57) and (58), most of the directions γ give the limit value $|\Omega_{\mathrm{cr}}^\pm(\gamma)| > \Omega_0^+$ with an exception for $\gamma=\gamma_*$ and $\gamma=-\gamma_*$, so that $\Omega_{\mathrm{cr}}^+(\gamma_*)=\Omega_0^+$ and $\Omega_{\mathrm{cr}}^-(-\gamma_*)=-\Omega_0^+$. This means that the critical value of the gyroscopic parameter Ω generally jumps up for infinitely small

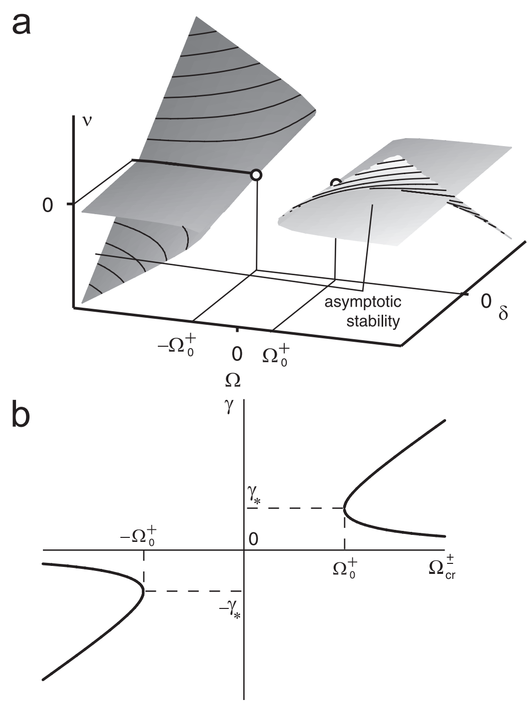

Fig. 9. Blowing the domain of gyroscopic stabilization of a statically unstable conservative system with $\mathbf{K} < 0$ up to the domain of asymptotic stability with the Whitney umbrella singularities (a). The limits of the critical gyroscopic parameter $\Omega_{\rm cr}^{\pm}$ as functions of $\gamma = \nu/\delta$ (b).

δ and ν, which is characteristic of the destabilization paradox in the Ziegler form. This effect illustrates high sensitivity of the critical parameters at the onset of flutter to small imperfections (an important example is the behavior of the critical angular velocity of a rotating disk of the squealing automotive brake [46]). As it is seen from the expression following from the characteristic polynomial (10)

$$\begin{aligned}\omega_{\rm cr}(\delta, \Omega, \nu) &= \pm\,\omega_0\sqrt{1 + 2\frac{\Omega\gamma - \Omega_0^+\gamma_*}{\omega_0^2\,{\rm tr}\,\mathbf{D}}}\\ &= \pm\,\omega_0 \pm \frac{\Omega_0^+(\gamma - \gamma_*) + \gamma_*(\Omega - \Omega_0^+)}{{\rm tr}\,\mathbf{D}}\\ &\quad + o(\gamma - \gamma_*, \Omega - \Omega_0^+), \qquad (59)\end{aligned}$$

the critical frequency of flutter also demonstrates an abrupt change due to the non-Hamiltonian perturbations of a gyroscopic system.

Therefore, in the presence of small damping and non-conservative positional forces, gyroscopic forces can both destabilize a statically stable conservative system (gyroscopic destabilization) and stabilize a statically unstable conservative system (gyroscopic stabilization). The first effect is essentially related with the dihedral angle singularity of the stability boundary, whereas the second one is governed by the Whitney umbrella singularity. In the next section we demonstrate how these singularities appear in the mechanical systems described by the modified Maxwell–Bloch equations [20,22,36,44,47].

4. The modified Maxwell–Bloch equations and its applications

The modified Maxwell–Bloch equations are the normal form for rotationally symmetric, planar dynamical systems [20,36,44]. They are a particular case of Eq. (1) for $m = 2$, $\mathbf{D} = \mathbf{I}$, and $\mathbf{K} = \kappa\mathbf{I}$, and thus can be written as a single differential equation with the complex coefficients

$$\ddot{x} + {\rm i}\Omega\dot{x} + \delta\dot{x} + {\rm i}\nu x + \kappa x = 0, \quad x = x_1 - {\rm i}x_2, \qquad (60)$$

where parameter κ corresponds to potential forces. This simple equation is important in studying some problems of gyrodynamics, such as tippe top inversion and the rising egg phenomena [22,36,44,47].

According to stability conditions (15)–(19) the zero solution of the modified Maxwell–Bloch equations is asymptotically stable if

$$\delta > 0, \quad \Omega > \frac{\nu}{\delta} - \frac{\delta}{\nu}\kappa, \qquad (61)$$

which also agrees with the Bilharz criterion applied to the complex polynomial $\lambda^2 + (\delta + {\rm i}\Omega)\lambda + \kappa + {\rm i}\nu$ [50].

Since the matrices $\mathbf{D}$ and $\mathbf{K}$ cannot be indefinite, then according to the results of the previous section the asymptotic stability domain of Eq. (60) has one of the two typical configurations, shown in Fig. 10. For $\kappa > 0$ the domain of asymptotic stability is a dihedral angle with the Ω-axis serving as its edge, Fig. 10(a). The sections of the domain by the planes $\Omega = {\rm const}$ are contained in the angle-shaped regions with the boundaries

$$\nu = \frac{\Omega \pm \sqrt{\Omega^2 + 4\kappa}}{2}\delta. \qquad (62)$$

The domain shown in Fig. 10(a) is a particular case of that depicted in Fig. 5(a). According to the Merkin theorem [10], for $\mathbf{K} = \kappa\mathbf{I}$ the interval of stability of a circulatory system $[-\nu_{\rm f}, \nu_{\rm f}]$ shown in Fig. 5(a) shrinks to a point so that at $\Omega = 0$ the angle is bounded by the lines $\nu = \pm\delta\sqrt{\kappa}$ and thus it is less than π. The domain of asymptotic stability is twisting around the Ω-axis in such a manner that it always remains in the half-space $\delta > 0$, Fig. 10(a). Consequently, the system stable at $\Omega = 0$ can become unstable at greater Ω, as shown in Fig. 10(a) by the dashed line. The larger magnitudes of circulatory forces, the lower $|\Omega|$ at the onset of instability.

When $\kappa > 0$ is decreasing, the hypersurfaces forming the dihedral angle move towards each other. At $\kappa = 0$ they are temporarily glued along the line $\nu = 0$ and for $\kappa < 0$ a new configuration is born, Fig. 10(b). The new domain of asymptotic stability consists of two non-intersecting parts given by the pockets of two Whitney umbrellas. The absolute values of the gyroscopic parameter Ω in the stability domain are always not less than $\Omega_0^+ = 2\sqrt{-\kappa}$. As a consequence, the system unstable at $\Omega = 0$ can become asymptotically stable at greater Ω, as shown in Fig. 10(b) by the dashed line.

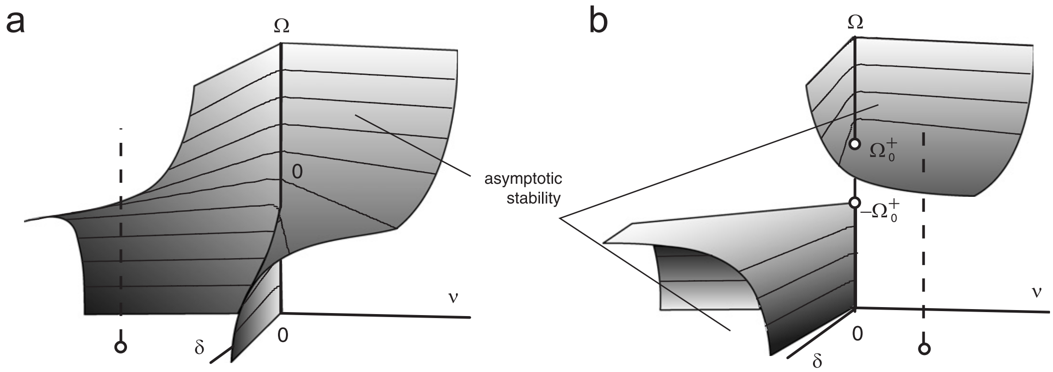

Fig. 10. Two configurations of the asymptotic stability domain of the modified Maxwell–Bloch equations for $\kappa > 0$ (a) and $\kappa < 0$ (b) corresponding to *gyroscopic destabilization* and *gyroscopic stabilization*, respectively.

4.1. The Crandall gyropendulum

The Crandall gyropendulum is an axisymmetric rigid body pivoted at a point O on the axis as shown in Fig. 11. When the axial spin γ is absent, the upright position is statically unstable. When γ is non-zero the body becomes a gyroscopic pendulum. Its primary parameters are its mass m, the distance L between the mass center and the pivot point, the axial moment of inertia I_a, and the diametral moment of inertia I_d about the pivot point; the gravity acceleration is denoted by g [22].

It is assumed that a drag force proportional to the linear velocity of the center of mass of the gyropendulum acts at the center of mass to oppose that velocity (stationary damping with the coefficient b_s). Additionally, it is assumed that a rigid sphere concentric with the pendulum tip O, is attached to the pendulum and rubs against a fixed rub plate. The gyropendulum is supported frictionlessly at O, while a viscous friction force acts between the larger sphere and the rub plate, being responsible for the rotating damping with the coefficient b_r. The linearized equations of motion for the gyropendulum in the vicinity of the vertical equilibrium position derived in [22] have the form (60) with the coefficients

$$\delta = \sigma + \rho, \quad \Omega = \eta\gamma, \quad \kappa = -\alpha^2, \quad \nu = \rho\gamma, \tag{63}$$

where parameters of the system are given by the expressions

$$\eta = \frac{I_a}{I_d}, \quad \sigma = \frac{b_s}{I_d}, \quad \rho = \frac{b_r}{I_d}, \quad \alpha^2 = \frac{mgL}{I_d}. \tag{64}$$

The parameter η is responsible for the shape of the gyropendulum: for $\eta < 1$ the pendulum is a rod-like, and for $\eta > 1$ it is a disk-like. The parameters σ and ρ correspond to the stationary and rotating damping, α is the non-spinning pendulum frequency.

According to expressions (61) and (63), the asymptotic stability domain of the Crandall gyropendulum is given by the conditions

$$\gamma > \gamma_{cr}^{+}(\rho, \sigma), \quad \gamma < \gamma_{cr}^{-}(\rho, \sigma), \quad \sigma + \rho > 0, \tag{65}$$

where the critical values of the spin γ as a function of the two damping parameters are

$$\gamma_{cr}^{\pm}(\rho, \sigma) = \pm \frac{(\sigma + \rho)\alpha}{\sqrt{-\rho^2 + \rho^2\eta + \rho\eta\sigma}}. \tag{66}$$

Eq. (66) describes two surfaces in the space of the parameters (ρ, σ, γ). Both surfaces have a singularity Whitney umbrella at the points $(0, 0, \pm\gamma_0^+)$, where $\gamma_0^+ = 2\sqrt{-\kappa}/\eta = 2\alpha/\eta$ is the critical spin of the undamped system ($\sigma = \rho = 0$). The surface $\gamma_{cr}^+(\rho, \sigma)$ is shown in Fig. 11 for $\alpha = 1$ and $\eta = 2$. The inequality $\sigma + \rho > 0$ selects the stable pocket of the Whitney umbrella. As it follows from expression (66), $\Omega_{cr}^+ \geqslant \Omega_0^+$ and $\Omega_{cr}^- \leqslant -\Omega_0^+$. The critical loads coincide only for the specific ratios of the coefficients of the stationary and rotating damping

$$\frac{b_s}{b_r} = \frac{\sigma}{\rho} = \frac{2-\eta}{\eta} = \frac{\Omega_0^+}{\omega_0} - 1, \tag{67}$$

where $\omega_0 = \alpha$ is the critical frequency of the undamped pendulum [22].

4.2. Rising egg

When one spins sufficiently fast a hard-boiled egg with its long axis horizontal, the egg rises from the horizontal state to a vertical state as shown in Fig. 12. This effect caused by the combined action of the dissipative, gyroscopic, and nonconservative positional forces is known as the rising egg phenomenon [44].

In [44] the egg of mass M is modelled by a prolate spheroid surface with its equatorial radius less than the polar radius R, so that their ratio $\alpha < 1$. The mass distribution is assumed to be symmetric, that is the center of mass coincides with the geometric center, the moments of inertia about the two principal axes in the equatorial plane are equal to I, and the moment of inertia about the axis of symmetry is I_k. The gravity acceleration is denoted by g. It is assumed that the surface frictional force exerted on the body at the contact point is proportional with the coefficient c to the velocity of the contact point on the rigid body relative to the center of mass [44].

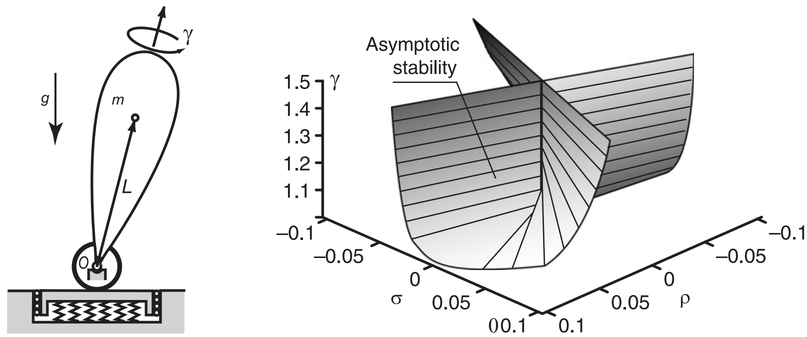

Fig. 11. The Crandall gyropendulum and its domain of asymptotic stability.

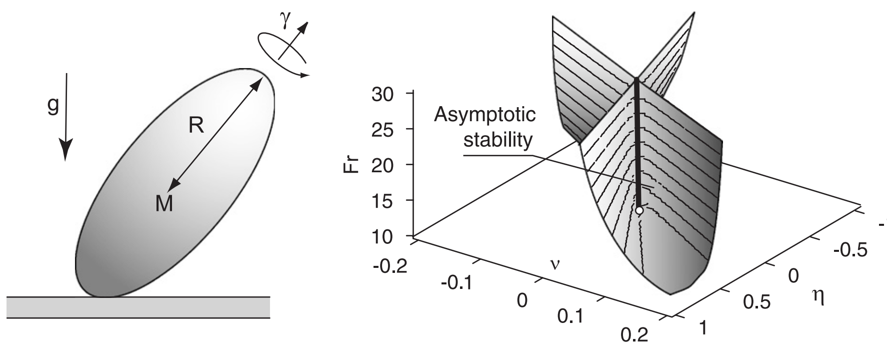

Fig. 12. The rising egg and the domain of asymptotic stability of its risen state.

With the dimensionless inertia ratio σ, Froude number Fr, mass μ, and friction factor η,

$$\sigma = \frac{I_k}{I}, \quad Fr = \frac{\gamma^2 R}{g}, \quad \mu = \frac{MR^2}{I}, \quad \eta = \frac{cR^2}{I\gamma}, \tag{68}$$

the equations of motion linearized about the risen state with the spin rate γ are in the form of the modified Maxwell–Bloch equations (60) with the coefficients

$$\delta = \eta, \quad \Omega = -\alpha, \quad \kappa = \mu\frac{\alpha^2 - 1}{Fr}, \quad \nu = \frac{\alpha^3\eta}{\sigma}. \tag{69}$$

Since $\alpha < 1$, the coefficient $\kappa < 0$ and in the absence of dissipation ($\eta=\nu=0$), the gyroscopic system is stable for $\Omega^2 > -4\kappa$, which is equivalent to the inequality

$$Fr > Fr^0 = \frac{4\mu(1-\alpha^2)}{\alpha^2}. \tag{70}$$

When dissipative and circulatory forces are acting, then, according to conditions (61) and expressions (69) the risen state is asymptotically stable if $\eta > 0$ and

$$Fr > Fr^{\rm cr} = Fr^0 \frac{\alpha^2\eta^2}{4\nu(\alpha\eta - \nu)} \geqslant Fr^0. \tag{71}$$

At the given α and μ the critical Froude number of the damped system $Fr^{\rm cr}$ is a function of the damping coefficient η and inertia ratio σ. However, due to the relation $\nu = \alpha^3\eta/\sigma$ we consider it as a function of the magnitudes of damping and circulatory forces $Fr^{\rm cr} = Fr^{\rm cr}(\eta, \nu)$.

The asymptotic stability domain of the risen state in the space of the parameters η, ν, and Fr is shown in Fig. 12 for $\mu = 1$ and $\alpha = \frac{1}{2}$. It has typical singular form implying that $Fr^{\rm cr} \geqslant Fr^0$, where the equality is attained only for $\nu = \eta\alpha/2$, which is equivalent to $\sigma = 2\alpha^2$.

4.3. Tippe top

The most common geometric form of tippe top is a cylindrical stem attached to a truncated ball [36,37]. On a flat surface, the tippe top rests stably with its stem up (*non-inverted state*). However, spun fast enough on its blunt end, the tippe top inverts, and spins on its stem (*inverted state*) [36].

In [36] the tippe top is modelled as a ball of radius R and mass M on a fixed plane, see Fig. 13 (left). The mass distribution of the ball is inhomogeneous, but symmetric about an axis through the ball's geometric center. Thus, the ball's center of mass is located on the axis of symmetry, but at a distance eR from the geometric center, where e, $|e| \leqslant 1$ is the center of mass offset. If in the non-inverted state the center of mass is above the geometric center, then $e > 0$; if below, $e < 0$. The moment of inertia about the axis of symmetry is denoted by I_k. The inertias about the two other principal axes attached to the center of the ball are equal to I. The gravity acceleration is g. It is assumed that the surface frictional force at the contact point is proportional with the coefficient c to the velocity of the contact point on the rigid body relative to the center of mass [36].

The equations of motion linearized about the non-inverted state with the spin rate γ are in the form of the modified

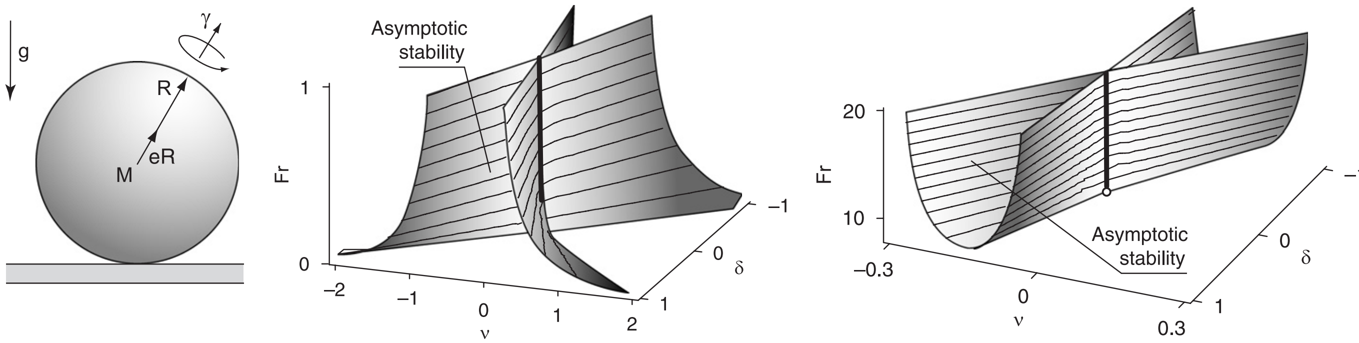

Fig. 13. The tippe top (left) and the domains of asymptotic stability of its non-inverted (center) and inverted (right) states.

Maxwell–Bloch equations (60) with the coefficients

$$\delta = -\frac{(1+e)^2\eta}{-1+e^2\mu}, \quad \Omega = \frac{\sigma}{-1+e^2\mu},$$
$$\kappa = \frac{Fr^{-1}e\mu}{-1+e^2\mu}, \quad \nu = \frac{\eta(1+e)}{-1+e^2\mu}, \tag{72}$$

where the dimensionless inertia ratio σ, Froude number Fr, mass μ, and friction coefficient η are

$$\sigma = \frac{I_k}{I}, \quad Fr = \frac{\gamma^2 R}{g}, \quad \mu = \frac{MR^2}{I}, \quad \eta = \frac{cR^2}{I\gamma}. \tag{73}$$

Linearization about the inverted state yields the modified Maxwell–Bloch equations (60) with the coefficients

$$\delta = -\frac{(1-e)^2\eta}{-1+e^2\mu}, \quad \Omega = \frac{\sigma(1+e-2\mu e^2)}{(1-e)(-1+e^2\mu)},$$
$$\kappa = \frac{-Fr^{-1}e\mu}{-1+e^2\mu}, \quad \nu = \frac{\eta(1+e)}{-1+e^2\mu}. \tag{74}$$

Since the eccentricity $|e|<1$ we restrict our subsequent considerations to the case $1-e^2\mu>0$. Then, in the absence of dissipation ($\eta=0$), the non-inverted state with $e<0$, is stable for all Froude numbers because $\kappa>0$. When damping forces are acting, then according to stability conditions (61) the domain of asymptotic stability of the non-inverted state is a dihedral angle in the space of parameters δ, ν, and Fr, as shown at the central picture in Fig. 13 for $\mu=1$, $e=-\frac{1}{5}$, and $\sigma=\frac{1}{2}$. The boundaries of the stability domain are given by the expressions

$$\nu = \frac{1}{2}\frac{Fr\,\sigma \pm \sqrt{(Fr\,\sigma)^2+4e\mu\,Fr\,(-1+e^2\mu)}}{Fr\,(-1+e^2\mu)}\delta. \tag{75}$$

Since $e<0$ and $1-e^2\mu>0$, the radicand in expressions (75) is always positive and for every Fr they define two lines intersecting at the origin, see Fig. 13. The angle between the lines is getting smaller with the increase in Fr. That means that statically stable non-inverted state becomes unstable for sufficiently large Froude number

$$Fr > \frac{\delta^2 e\mu}{\nu(-\sigma\delta+\nu(-1+e^2\mu))}. \tag{76}$$

As it has been shown in [36] there exists a heteroclinic connection between the non-inverted and inverted states of tippe top. Thus, the non-inverted state which has lost its stability, can be transferred to the inverted state. In the absence of dissipation the inverted state is stable for $\Omega^2>-4\kappa$, which is equivalent to the inequality

$$Fr > Fr^0 := \frac{4\mu e(1-e)^2(-1+\mu e^2)}{\sigma^2(1+e-2\mu e^2)^2}. \tag{77}$$

We note that $Fr^0>0$ because $e<0$ and $1-e^2\mu>0$. If the damping is taken into account then according to stability conditions (61) the inverted state is asymptotically stable for $\delta>0$ and

$$Fr > Fr^{\mathrm{cr}} := Fr^0 \frac{\delta^2\sigma^2(1+e-2\mu e^2)^2}{4\nu(1-e)(-1+\mu e^2)(\sigma\delta(1+e-2\mu e^2)+\nu(1-\mu e^2)(1-e))} \geqslant Fr^0, \tag{78}$$

where the equality is attained only at

$$\nu = \left(\frac{\sigma}{2(1-\mu e^2)} - \frac{\sigma}{1-e}\right)\delta. \tag{79}$$

The domain of asymptotic stability of the inverted state is shown in the right picture of Fig. 13 for $\mu=1$, $e=-\frac{1}{5}$, and $\sigma=\frac{1}{2}$. It has a recognizable form of the half of the Whitney umbrella.

5. Conclusions

For a general linear mechanical system with two degrees of freedom the effect of small damping and non-conservative positional forces on the stability of a gyroscopic system as well as the effect of small gyroscopic and damping forces on the stability of a circulatory system has been studied.

It was found that the stability boundary of both the imperfect Hamiltonian system and the imperfect reversible one possesses singularities such as Whitney umbrella, and dihedral and trihedral angles. Dihedral angle singularity is responsible for the loss of stability by a gyroscopic system, which is statically stable in the absence of gyroscopic forces, due to action of the small damping and circulatory forces. Whitney umbrella singularity is the reason for the destabilization paradox appearing both in the circulatory systems with weak velocity-dependent forces and in the gyroscopically stabilized but statically unstable conservative systems perturbed by small damping and circulatory forces. The Crandall gyropendulum, rising egg, and tippe top, considered as mechanical examples, confirm the conclusions of the theory.

As it has been noted by Sevryuk [2] there is a very close similarity between the behavior of solutions of reversible systems and that of Hamiltonian ones. The destabilization paradox due to breaking the Hamiltonian and reversible symmetry remarkably reveals both the similarity and difference between the two types of systems.

Acknowledgments

The author thanks Peter Hagedorn and the Department of Mechanical Engineering of the Technical University of Darmstadt where the research for this paper has been done during the Alexander von Humboldt Fellowship.

References

[1] V.N. Tkhai, On stability of mechanical systems under the action of position forces, Prikl. Mat. Mekh. USSR 44 (1981) 24–29.
[2] M.B. Sevryuk, Reversible Systems, Lecture Notes in Mathematics, vol. 1211, Springer, Berlin, 1986.
[3] V.I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations, Springer, New York, Berlin, 1983.
[4] W. Thomson, P.G. Tait, Treatise on Natural Philosophy, vol. 1, Part 1, New Edition, Cambridge University Press, Cambridge, 1879.
[5] E.J. Routh, A Treatise on the Stability of a Given State of Motion, Macmillan, London, 1892.
[6] D.M. Smith, The motion of a rotor carried by a flexible shaft in flexible bearings, Proc. Roy. Soc. London A 142 (1933) 92–118.
[7] H. Ziegler, Die Stabilitatskriterien der Elastomechanik, Ing.-Arch. 20 (1952) 49–56.
[8] V.V. Bolotin, Non-conservative Problems of the Theory of Elastic Stability, Pergamon, Oxford, 1963.
[9] G. Herrmann, I.C. Jong, On the destabilizing effect of damping in nonconservative elastic systems, ASME J. Appl. Mech. 32 (3) (1965) 592–597.
[10] D.R. Merkin, Introduction to the Theory of Stability, Springer, Berlin, 1997.
[11] V.M. Lakhadanov, On stabilization of potential systems, Prikl. Mat. Mekh. 39 (1) (1975) 53–58.
[12] A.V. Karapetyan, On the stability of nonconservative systems, Vestn. Mosk. Univ. Ser. Mat. Mekh. 4 (1975) 109–113.
[13] I.P. Andreichikov, V.I. Yudovich, The stability of visco-elastic rods, Izv. Acad. Nauk SSSR, Mekh. Tverd. Tela 1 (1975) 150–154.
[14] N.V. Banichuk, A.S. Bratus, A.D. Myshkis, Stabilizing and destabilizing effects in nonconservative systems, Prikl. Mat. Mekh. USSR 53 (2) (1989) 158–164.
[15] A.P. Seyranian, Destabilization paradox in stability problems of non-conservative systems, Adv. Mech. 13 (2) (1990) 89–124.
[16] R.S. MacKay, Movement of eigenvalues of Hamiltonian equilibria under non-Hamiltonian perturbation, Phys. Lett. A 155 (1991) 266–268.
[17] A.N. Kounadis, On the paradox of the destabilizing effect of damping in nonconservative systems, Int. J. Nonlinear Mech. 27 (1992) 597–609.
[18] G. Haller, Gyroscopic stability and its loss in systems with two essential coordinates, Int. J. Nonlinear Mech. 27 (1992) 113–127.
[19] V.F. Zhuravlev, Nutational vibrations of a free gyroscope, Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela 6 (1992) 13–16.
[20] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, T.S. Ratiu, Dissipation induced instabilities, Ann. Inst. Henri Poincaré 11 (1) (1994) 37–90.
[21] J. Maddocks, M.L. Overton, Stability theory for dissipatively perturbed Hamiltonian systems, Commun. Pure Appl. Math. 48 (1995) 583–610.
[22] S.H. Crandall, The effect of damping on the stability of gyroscopic pendulums, Z. Angew. Math. Phys. 46 (1995) S761–S780.
[23] V.V. Beletsky, Some stability problems in applied mechanics, Appl. Math. Comput. 70 (1995) 117–141.
[24] A.P. Seyranian, J. Stoustrup, W. Kliem, On gyroscopic stabilization, Z. Angew. Math. Phys. 46 (1995) 255–267.
[25] O.M. O'Reilly, N.K. Malhotra, N.S. Namachchivaya, Some aspects of destabilization in reversible dynamical systems with application to follower forces, Nonlinear Dyn. 10 (1996) 63–87.
[26] A.P. Seyranian, On stabilization of non-conservative systems by dissipative forces and uncertainty of critical load, Dokl. Akad. Nauk. 348 (3) (1996) 323–326 (in Russian).
[27] P. Freitas, M. Grinfeld, P.A. Knight, Stability of finite-dimensional systems with indefinite damping, Adv. Math. Sci. Appl. 7 (1) (1997) 437–448.
[28] A.A. Mailybaev, A.P. Seyranian, On singularities of a boundary of the stability domain, SIAM J. Matrix Anal. Appl. 21 (1) (1999) 106–128.
[29] V.V. Bolotin, A.A. Grishko, M.Yu. Panov, Effect of damping on the postcritical behavior of autonomous non-conservative systems, Int. J. Nonlinear Mech. 37 (2002) 1163–1179.
[30] K. Popp, M. Rudolph, M. Kröger, M. Lindner, Mechanisms to generate and to avoid friction induced vibrations, VDI-Berichte, vol. 1736, VDI-Verlag, Düsseldorf, 2002, pp. 1–15.
[31] N. Hoffmann, L. Gaul, Effects of damping on mode-coupling instability in friction induced oscillations: imperfect merging of modes and viscous instability, Z. Angew. Math. Mech. 83 (2003) 524–534.
[32] O.N. Kirillov, How do small velocity-dependent forces (de)stabilize a non-conservative system? DCAMM Report, vol. 681, 2003.
[33] A.P. Seyranian, O.N. Kirillov, Effect of small dissipative and gyroscopic forces on the stability of nonconservative systems, Dokl. Phys. 48 (12) (2003) 679–684.
[34] W.F. Langford, Hopf meets Hamilton under Whitney's umbrella, in: Nonlinear Stochastic Dynamics; Proceedings of the IUTAM Symposium, 2003, pp. 157–167.
[35] A.P. Ivanov, The stability of mechanical systems with positional non-conservative forces, J. Appl. Math. Mech. 67 (5) (2003) 625–629.
[36] N.M. Bou-Rabee, J.E. Marsden, L.A. Romero, Tippe top inversion as a dissipation-induced instability, SIAM J. Appl. Dyn. Syst. 3 (2004) 352–377.
[37] H.K. Moffatt, Y. Shimomura, M. Branicki, Dynamics of an axisymmetric body spinning on a horizontal surface. I. Stability and the gyroscopic approximation, Proc. Roy. Soc. London A 460 (2004) 3643–3672.
[38] O.N. Kirillov, Destabilization paradox, Dokl. Phys. 49 (4) (2004) 239–245.
[39] O.N. Kirillov, A.P. Seyranian, Stabilization and destabilization of a circulatory system by small velocity-dependent forces, J. Sound Vib. 283 (3–5) (2005) 781–800.
[40] O.N. Kirillov, A theory of the destabilization paradox in non-conservative systems, Acta Mech. 174 (3–4) (2005) 145–166.
[41] O.N. Kirillov, A.P. Seyranian, Instability of distributed nonconservative systems caused by weak dissipation, Dokl. Math. 71 (3) (2005) 470–475.
[42] O.N. Kirillov, A.O. Seyranian, The effect of small internal and external damping on the stability of distributed non-conservative systems, J. Appl. Math. Mech. 69 (4) (2005) 529–552.

[43] O.N. Kirillov, Sensitivity analysis of the roots of the characteristic polynomial and stability of non-conservative systems, in: Proceedings of the International Conference "Physics and Control", St. Petersburg, Russia, 2005, pp. 206–211.
[44] N.M. Bou-Rabee, J.E. Marsden, L.A. Romero, A geometric treatment of Jellet's egg, Z. Angew. Math. Mech. 85 (9) (2005) 618–642.
[45] A.N. Kounadis, Hamiltonian weakly damped autonomous systems exhibiting periodic attractors, Z. Angew. Math. Phys. 57 (2006) 324–350.
[46] U. von Wagner, D. Hochlenert, P. Hagedorn, Minimal models for disk brake squeal, J. Sound Vib., 2007, in press, doi:10.1016/j.jsv.2006.11.023.
[47] R. Krechetnikov, J.E. Marsden, On destabilizing effects of two fundamental non-conservative forces, Physica D 214 (2006) 25–32.
[48] O.N. Kirillov, Gyroscopic stabilization of non-conservative systems, Phys. Lett. A 359 (3) (2006) 204–210.
[49] S. Barnett, Leverrier's algorithm: a new proof and extensions, SIAM J. Matrix Anal. Appl. 10 (4) (1989) 551–556.
[50] H. Bilharz, Bemerkung zu einem Satze von Hurwitz, Z. Angew. Math. Mech. 24 (2) (1944) 77–82.
[51] I. Hoveijn, M. Ruijgrok, The stability of parametrically forced coupled oscillators in sum resonance, Z. Angew. Math. Phys. 46 (1995) 384–392.