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Gyroscopic stabilization of non-conservative systems

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Abstract

Gyroscopic stabilization of a linear conservative system, which is statically unstable, can be either improved or destroyed by weak damping and circulatory forces. This is governed by Whitney umbrella singularity of the boundary of the asymptotic stability domain of the perturbed system.

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1. Introduction

The system under consideration is described by a linear differential equation of second order with the matrix coefficients

$$\ddot{\mathbf{y}} + (\delta\mathbf{D} + \gamma\mathbf{G})\dot{\mathbf{y}} + (\mathbf{K} + \nu\mathbf{N})\mathbf{y} = 0, \tag{1}$$

where dot denotes time differentiation, $\mathbf{y} \in \mathbb{R}^m$, and real matrix $\mathbf{K} = \mathbf{K}^T$ corresponds to potential forces. Real matrices $\mathbf{D} = \mathbf{D}^T$, $\mathbf{G} = -\mathbf{G}^T$, and $\mathbf{N} = -\mathbf{N}^T$ are related to dissipative (damping), gyroscopic, and non-conservative positional (circulatory) forces with magnitudes controlled by the parameters δ, γ, and ν, respectively. In two important limiting cases when either damping and gyroscopic forces or damping and circulatory forces are absent, system (1) can be only marginally stable having its spectrum on the imaginary axis of the complex plane. In many engineering and physical applications it is vital to know how the marginal stability is improved or destroyed when these forces are taken into account [1–19]. The reason for the *destabilization paradox* in a circulatory system perturbed by small velocity-dependent forces [3,4,12] is the existence of the *Whitney umbrella* singularity [20] on its asymptotic stability boundary [11,15,17]. In this Letter we show that the stability boundary of system (1) with weak damping and circulatory forces possesses the same singularity, which strongly influences the stability of gyroscopic systems with friction in contact [2,7,8,13,14,18].

2. A gyroscopic system with weak damping and circulatory forces

We will restrict our subsequent considerations to the case when system (1) has only $m = 2$ degrees of freedom. Then, the matrices of gyroscopic and circulatory forces have the form $\mathbf{G} = \mathbf{N} = \mathbf{J}$, where

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

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The matrices of damping and potential forces $\mathbf{D}$ and $\mathbf{K}$ are assumed to be given and fixed.

We notice that the study of the two-degrees-of-freedom system (1) makes sense because it contains important low-dimensional models of actual dynamical systems, for example, the modified Maxwell–Bloch equations which are the normal form for rotationally symmetric, planar dynamical systems and which describe tippe top inversion [7,8,13,14,18,19]. Moreover, the qualitative conclusions obtained for two degrees of freedom remain valid in the general case, which can be investigated by perturbation approach developed in [5–8,11,15–17].

To study the stability of system (1) we consider its characteristic polynomial in the form given by the Leverrier–Barnett algorithm [21]

$$P(\lambda,\delta,\nu,\gamma)=\lambda^4+\delta\,\mathrm{tr}\,\mathbf{D}\lambda^3+\left(\mathrm{tr}\,\mathbf{K}+\delta^2\det\mathbf{D}+\gamma^2\right)\lambda^2+\left(\delta(\mathrm{tr}\,\mathbf{K}\,\mathrm{tr}\,\mathbf{D}-\mathrm{tr}\,\mathbf{KD})+2\gamma\nu\right)\lambda+\det\mathbf{K}+\nu^2. \tag{3}$$

We will concentrate on the case when system (1) is close to the conservative gyroscopic system

$$\ddot{\mathbf{y}}+\gamma\mathbf{G}\dot{\mathbf{y}}+\mathbf{K}\mathbf{y}=0. \tag{4}$$

The values of the parameters δ and ν are assumed to be small, while the parameter γ can be arbitrary large. Our goal is to find and analyze the asymptotic stability domain of system (1) in the space of the parameters δ, ν, and γ.

In the absence of the damping and circulatory forces ($\delta=\nu=0$) the characteristic polynomial (3) has four roots $-\lambda_+$, $-\lambda_-$, λ_-, and λ_+, where

$$\lambda_\pm=\sqrt{-\frac{1}{2}\left(\mathrm{tr}\,\mathbf{K}+\gamma^2\right)\pm\frac{1}{2}\sqrt{\left(\mathrm{tr}\,\mathbf{K}+\gamma^2\right)^2-4\det\mathbf{K}}}. \tag{5}$$

At $\gamma=0$ system (4) is conservative. It is stable when the eigenvalues (5) are purely imaginary, which happens if $\mathrm{tr}\,\mathbf{K}>0$ and $\det\mathbf{K}>0$ implying positive-definiteness of the 2×2 symmetric matrix $\mathbf{K}$. Otherwise, the system is statically unstable (divergence) due to the existence of positive real eigenvalues.

If $\gamma\neq 0$, the eigenvalues (5) can be real (divergence), complex (dynamic instability or flutter) or purely imaginary (stability). It is well known that for $\det\mathbf{K}>0$ and $\mathrm{tr}\,\mathbf{K}>0$ system (4) is stable at any γ; for $\det\mathbf{K}<0$ the gyroscopic system is statically unstable [1]. In the case when $\det\mathbf{K}>0$ and $\mathrm{tr}\,\mathbf{K}<0$, which is equivalent to negative definiteness of the 2×2 matrix $\mathbf{K}$, the *gyroscopic stabilization* of the statically unstable conservative system is possible, see e.g. [8]. Indeed, let us write Eq. (5) in the form

$$\lambda_\pm=\sqrt{-\frac{1}{2}\left(\gamma^2-\frac{1}{2}\left(\gamma_0^{-2}+\gamma_0^{+2}\right)\right)\pm\frac{1}{2}\sqrt{\left(\gamma^2-\gamma_0^{-2}\right)\left(\gamma^2-\gamma_0^{+2}\right)}}, \tag{6}$$

where the critical values $\gamma_0^\pm$ of the gyroscopic parameter γ are given by the expressions

$$0<\sqrt{-\mathrm{tr}\,\mathbf{K}-2\sqrt{\det\mathbf{K}}}=:\gamma_0^-\leqslant\gamma_0^+:=\sqrt{-\mathrm{tr}\,\mathbf{K}+2\sqrt{\det\mathbf{K}}}. \tag{7}$$

At $\gamma=0$ there are in general four real roots $\pm\lambda_\pm=\pm(\gamma_0^+\pm\gamma_0^-)/2$ and system (4) is statically unstable. With the increase of γ^2 the distance $\lambda_+-\lambda_-$ between two roots of the same sign is getting smaller. The roots are moving towards each other until they merge at $\gamma^2=\gamma_0^{-2}$ with the origination of a pair of double real eigenvalues $\pm\omega_0$ with the Jordan blocks, where

$$\omega_0=\frac{1}{2}\sqrt{\gamma_0^{+2}-\gamma_0^{-2}}=\sqrt[4]{\det\mathbf{K}}>0. \tag{8}$$

Further increase of γ^2 yields splitting of $\pm\omega_0$ to two couples of complex conjugate eigenvalues lying on the circle

$$\mathrm{Re}\,\lambda^2+\mathrm{Im}\,\lambda^2=\omega_0^2. \tag{9}$$

The complex eigenvalues move along the circle until at $\gamma^2=\gamma_0^{+2}$ they reach the imaginary axis and originate a complex-conjugate pair of double purely imaginary eigenvalues $\pm i\omega_0$. For $\gamma^2>\gamma_0^{+2}$ the double eigenvalues split into four simple purely imaginary eigenvalues which do not leave the imaginary axis, Fig. 1.

Thus, system (4) with $\mathbf{K}<0$ is statically unstable for $\gamma\in(-\gamma_0^-,\gamma_0^-)$, it is dynamically unstable for $\gamma\in[-\gamma_0^+,-\gamma_0^-]\cup[\gamma_0^-,\gamma_0^+]$, and it is stable (gyroscopic stabilization) for $\gamma\in(-\infty,-\gamma_0^+)\cup(\gamma_0^+,\infty)$, see Fig. 1. The values of the gyroscopic parameter $\pm\gamma_0^-$ define the boundary between the divergence and flutter domains while the values $\pm\gamma_0^+$ define the flutter-stability boundary. In the following, we investigate how do small damping and circulatory forces blow-up the stability domain of conservative gyroscopic system (4) with $\mathbf{K}<0$ in the space of the parameters δ, ν, and γ.

As it has been established in [20], the boundary of the asymptotic stability domain of a multiparameter family of real matrices is not a smooth surface. Generically, it possesses singularities corresponding to multiple eigenvalues with zero real part. In particular, for real matrices depending on three parameters, two different pairs of simple purely imaginary eigenvalues originate a singularity

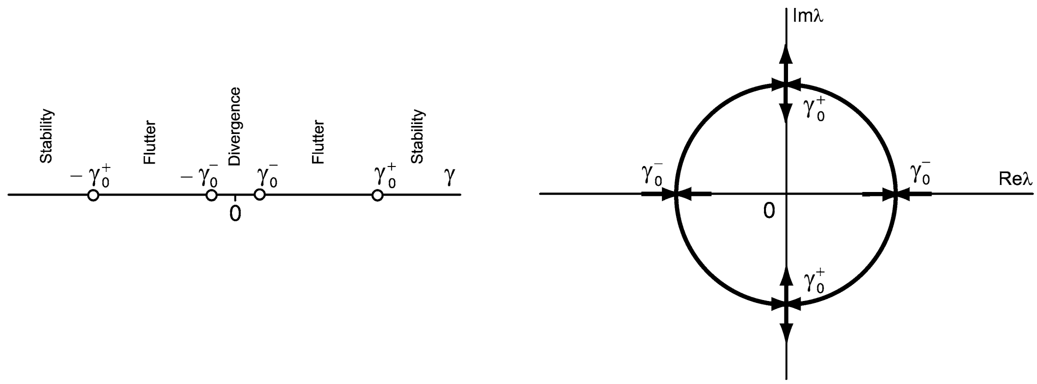

Fig. 1. Stability diagram of a conservative gyroscopic system with $\mathbf{K} < 0$ (left) and the corresponding trajectories of the eigenvalues in the complex plane for the increasing parameter $\gamma > 0$ (right).

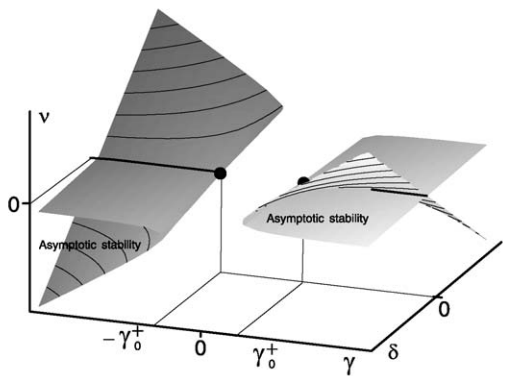

Fig. 2. Whitney umbrella singularities on the stability boundary of the non-conservative system (1) with $\mathbf{K} < 0$. The arms of the umbrellas are exactly stability domains of a conservative gyroscopic system, see Fig. 1. The condition $\delta \operatorname{tr} \mathbf{D} > 0$ selects the stable pockets of the umbrellas (deadlocks of an edge).

of the stability boundary, which is shaped as a dihedral angle in the parameter space. A pair of double purely imaginary eigenvalues with the Jordan block corresponds to the singularity *deadlock of an edge*, which is a half of the Whitney umbrella surface [20].

Considering the asymptotic stability domain of system (1) in the space of the parameters δ, ν and γ we know that the γ-axis corresponds to the unperturbed conservative gyroscopic system. The parts of this axis that belong to the stability domain of system (4) and correspond to two different pairs of simple purely imaginary eigenvalues, form edges of dihedral angles, bounding the asymptotic stability domain of system (1). At the points $\pm\gamma_0^+$ of the γ-axis, corresponding to the stability-flutter boundary of system (4) there exists a pair of double purely imaginary eigenvalues with the Jordan block. Qualitatively, the asymptotic stability domain of system (1) in the space (δ, ν, γ) near the γ-axis looks like a dihedral angle which becomes more acute while approaching the points $\pm\gamma_0^+$. At these points the angle shrinks forming the deadlock of an edge, see Fig. 2.

Below with the use of the Routh–Hurwitz criterion we find exact expressions for the asymptotic stability domain in the space of the parameters δ, ν, and γ as well as its first-order approximations in the vicinity of the γ-axis, and show that the latter are reduced to the canonical equation of the Whitney umbrella.

The Routh–Hurwitz criterion in the form of Liénard and Chipart [22] for polynomial (3) with $\det \mathbf{K} > 0$ is presented by the inequalities

$$\delta \operatorname{tr} \mathbf{D} > 0, \tag{10}$$

$$\operatorname{tr} \mathbf{K} + \delta^2 \det \mathbf{D} + \gamma^2 > 0, \tag{11}$$

$$-a(\delta,\gamma)\nu^2 + 2\delta\gamma b(\delta,\gamma)\nu + \delta^2 c(\delta,\gamma) > 0, \tag{12}$$

where

$$\begin{aligned}
a(\delta,\gamma) &= 4\gamma^2 + \delta^2 \operatorname{tr} \mathbf{D}^2,\\
b(\delta,\gamma) &= \gamma^2 \operatorname{tr} \mathbf{D} - \operatorname{tr} \mathbf{K} \operatorname{tr} \mathbf{D} + 2 \operatorname{tr} \mathbf{KD} + \delta^2 \operatorname{tr} \mathbf{D} \det \mathbf{D},\\
c(\delta,\gamma) &= (\operatorname{tr} \mathbf{K} \operatorname{tr} \mathbf{D} - \operatorname{tr} \mathbf{KD})\big(\operatorname{tr} \mathbf{KD} + \delta^2 \operatorname{tr} \mathbf{D} \det \mathbf{D} + \gamma^2 \operatorname{tr} \mathbf{D}\big) - \det \mathbf{K} \operatorname{tr} \mathbf{D}^2.
\end{aligned} \tag{13}$$

Solving the quadratic equation we write stability condition (12) in the form

$$\big(\nu-\nu^-\delta\big)\big(\nu-\nu^+\delta\big)<0,\quad \nu^\pm(\delta,\gamma)=\frac{\gamma b(\delta,\gamma)\pm\sqrt{\gamma^2 b(\delta,\gamma)^2+a(\delta,\gamma)c(\delta,\gamma)}}{a(\delta,\gamma)}. \tag{14}$$

Conditions (10), (11), and (14) define the asymptotic stability domain of system (1) in the space (δ,ν,γ).

Let us consider the asymptotic stability domain in the plane (δ,ν) in the vicinity of the origin. From the structure of the coefficients (13) and expressions (5)–(7) it follows that the Taylor expansions of the functions $\nu^\pm(\delta,\gamma)$ for the fixed $\gamma\neq 0$ and small δ have the form $\nu^\pm(\delta,\gamma)=\nu_0^\pm(\gamma)+O(\delta^2)$ with

$$\nu_0^\pm(\gamma)=\frac{2\operatorname{tr}\mathbf{KD}+\operatorname{tr}\mathbf{D}\big(\gamma^2+\frac{1}{2}(\gamma_0^{-2}+\gamma_0^{+2})\pm\sqrt{(\gamma^2-\gamma_0^{-2})(\gamma^2-\gamma_0^{+2})}\big)}{4\gamma}. \tag{15}$$

The coefficients $\nu_0^\pm$ are real for $\gamma^2\geqslant\gamma_0^{+2}$. In this case, the domain given by the conditions (14) is approximated by a cone in the vicinity of the origin in the plane (δ,ν)

$$\big\{\nu_0^-\delta<\nu<\nu_0^+\delta\big\}\cup\big\{\nu_0^+\delta<\nu<\nu_0^-\delta\big\}. \tag{16}$$

For γ tending to infinity, the angle of the cone is expanding to $\pi/2$, while for γ approaching the critical values $\pm\gamma_0^+$ it is becoming more acute, and at $\gamma=\pm\gamma_0^+$ it shrinks to a line

$$\nu=\pm\frac{(\gamma_0^{+2}-\omega_0^2)\operatorname{tr}\mathbf{D}+\operatorname{tr}\mathbf{KD}}{2\gamma_0^+}\delta. \tag{17}$$

Assuming for simplicity $\det\mathbf{D}\geqslant 0$, we get

$$\gamma^2\geqslant\gamma_0^{+2}=-\operatorname{tr}\mathbf{K}+2\sqrt{\det\mathbf{K}}>-\operatorname{tr}\mathbf{K}>-\operatorname{tr}\mathbf{K}-\delta^2\det\mathbf{D}, \tag{18}$$

and condition (11) is satisfied. Then, the stable skirt of the cone (16) is selected by the inequality (10). Returning to the three-dimensional picture, one sees that conditions (16) define two hypersurfaces intersecting along the γ-axis, so that the angle between them at every particular value of γ is getting smaller while approaching the points $\pm\gamma_0^+$, as shown in Fig. 2. The parts of this surface lying in the half-space $\delta\operatorname{tr}\mathbf{D}>0$ are a linear approximation of the boundary of the asymptotic stability domain.

The critical value of the gyroscopic parameter $\gamma_{cr}(\delta,\nu)$ at which the non-conservative system (1) loses stability can deviate significantly from that of the conservative gyroscopic system (γ_0^+). To get an estimate of $\gamma_{cr}(\delta,\nu)$ we consider the formulas (15) and (16) in the vicinity of the point ($\delta=0,\nu=0,\gamma=\gamma_0^+$). Leaving only the terms which are constant or proportional to $\sqrt{\Delta\gamma}=\sqrt{\gamma-\gamma_0^+}$ in both the numerator and denominator of the expression (15) we write the inequality (16) in the form

$$\gamma>\gamma_{cr}^+(\delta,\nu),\quad \gamma_{cr}^+(\delta,\nu)=\gamma_0^++2\gamma_0^+\left[\frac{((\gamma_0^{+2}-\omega_0^2)\operatorname{tr}\mathbf{D}+\operatorname{tr}\mathbf{KD})\delta-2\gamma_0^+\nu}{2\delta\omega_0\gamma_0^+\operatorname{tr}\mathbf{D}}\right]^2. \tag{19}$$

It is remarkable that the expression for γ_{cr}^+ has the form $Z=X^2/Y^2$, which is a canonical equation for the singular surface known as the Whitney umbrella [20]. Expression (19) explicitly shows that the function $\gamma_{cr}^+(\delta,\nu)$ is non-differentiable at the origin and depends only on the ratio $\beta=\nu/\delta$. Therefore, the limit of $\gamma_{cr}^+(\delta,\nu)$ at the origin is not defined and strongly depends on the direction of approaching given by β. Most of the directions β give the limit value $\gamma_{cr}^+(\beta)>\gamma_0^+$. The latter means that the critical "angular velocity" γ generally jumps up for infinitely small δ and ν. Such "jumps" illustrate high sensitivity of the critical parameters responsible for the onset of the flutter instability (in particular, the squeal of a rotating disk in an automotive brake [18]) to small imperfections in non-conservative gyroscopic systems.

We have obtained expressions (15) and (19) by direct analysis of the Routh–Huritz conditions, which is effective only in low dimensions of the state space. More general way is to use methods based on eigenvalue perturbation [5–8,11,15–17]. Indeed, perturbing the gyroscopic system (4) by small damping and circulatory forces yields an increment to purely imaginary eigenvalue $i\omega(\gamma)$

$$\lambda=i\omega-\frac{i\omega\mathbf{u}^*\mathbf{D}\mathbf{u}\delta+\mathbf{u}^*\mathbf{J}\mathbf{u}\nu}{2i\omega\mathbf{u}^*\mathbf{u}}+o(\delta,\nu), \tag{20}$$

where $\mathbf{u}$ is an eigenvector corresponding to $i\omega$

$$\mathbf{u}(\gamma)=C\begin{pmatrix}-i\omega\gamma-k_{12}\\ -\omega^2+k_{11}\end{pmatrix}, \tag{21}$$

C is a complex coefficient, and the asterisk denotes Hermitian conjugate.

Since **D** is a real symmetric matrix and **J** is a real skew-symmetric matrix, the first order correction to $i\omega(\gamma)$ is a real quantity. Consequently, the expression

$$\nu = -i\omega\frac{\mathbf{u}^*\mathbf{D}\mathbf{u}}{\mathbf{u}^*\mathbf{J}\mathbf{u}}\delta = \frac{d_{11}(\omega^2\gamma^2 + k_{12}^2) + 2d_{12}k_{12}(\omega^2 - k_{11}) + d_{22}(\omega^2 - k_{11})^2}{2\gamma(\omega^2 - k_{11})}\delta$$
$$= \frac{d_{11}(\omega^2 - k_{11})(\omega^2 - k_{22}) + 2d_{12}k_{12}(\omega^2 - k_{11}) + d_{22}(\omega^2 - k_{11})^2}{2\gamma(\omega^2 - k_{11})}\delta = \frac{\operatorname{tr}\mathbf{KD} - \operatorname{tr}\mathbf{K}\operatorname{tr}\mathbf{D} + \operatorname{tr}\mathbf{D}\omega^2}{2\gamma}\delta, \tag{22}$$

gives a linear approximation to the domain of asymptotic stability in the plane (δ, ν) for arbitrary γ. Taking into account formulas (6) and (7), we transform expression (22) to the form $\nu = \nu_0^{\pm}\delta$, where the coefficients $\nu_0^{\pm}$ are given by Eq. (15), which implies the estimate (19). Note that in higher dimensions of the state space, exact expressions for eigenvalues and eigenvectors of the unperturbed gyroscopic system such as (6) and (21) usually cannot be found, and eigenvalue perturbation methods give only estimates like (19), see [15,17].

Thus, the surfaces $\nu = \nu^+(\delta,\gamma)\delta$ and $\nu = \nu^-(\delta,\gamma)\delta$ in the vicinity of the points $(0, 0, \pm\gamma_0^+)$ are represented by two Whitney umbrellas. The arms of the umbrellas are the intervals of the γ-axis $(-\infty, -\gamma_+)$ and (γ_+, ∞) which are stability domains of conservative gyroscopic system (4). The domains of the gyroscopic stabilization of non-conservative system (1) with small damping and circulatory forces are given by the pockets of the two Whitney umbrellas, as shown in Fig. 2.

3. Stability of a gyropendulum with stationary and rotating damping

As an example we consider the Crandall gyropendulum [8]. The pendulum is an axisymmetric rigid body pivoted at a point O on the axis as shown in Fig. 3. When the axial spin Ω is absent, the upright position is statically unstable. When $\Omega \neq 0$ the body becomes a gyroscopic pendulum. Its primary parameters are its mass m, the distance L between the mass center and the pivot point, the axial moment of inertia I_a, and the diametral moment of inertia I_d about the pivot point. The gravity acceleration is denoted by g.

It is assumed that a drag force proportional to the linear velocity of the center of mass of the gyropendulum acts at the center of mass to oppose that velocity (stationary damping with the coefficient b_s). Additionally, it is assumed that a rigid sphere concentric with the pendulum tip O, is attached to the pendulum and rubs against a fixed rub plate. The gyropendulum is supported frictionlessly at O, while a viscous friction force acts between the larger sphere and the rub plate, being responsible for the rotating damping with the coefficient b_r. The linearized equations of motion for the gyropendulum in the vicinity of the vertical equilibrium position derived in [8] have the form (1) with the matrices **G**, **D**, **K**, and **N** given by the expressions

$$\gamma\mathbf{G} = \begin{pmatrix} 0 & \eta\Omega \\ -\eta\Omega & 0 \end{pmatrix}, \qquad \delta\mathbf{D} = \begin{pmatrix} \sigma+\rho & 0 \\ 0 & \sigma+\rho \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix}, \qquad \nu\mathbf{N} = \begin{pmatrix} 0 & \rho\Omega \\ -\rho\Omega & 0 \end{pmatrix}. \tag{23}$$

The system depends on the spin Ω and four parameters

$$\eta = \frac{I_a}{I_d}, \qquad \sigma = \frac{b_s}{I_d}, \qquad \rho = \frac{b_r}{I_d}, \qquad \alpha^2 = \frac{mgL}{I_d}, \tag{24}$$

where α is the non-spinning pendulum frequency and η is responsible for the shape of the gyropendulum: for $\eta < 1$ the pendulum is prolate, and for $\eta > 1$ it is oblate. Parameters σ and ρ correspond to the stationary and rotating damping respectively. We notice that the stationary damping contributes only to the matrix $\delta\mathbf{D}$ while the rotating damping is responsible also for the appearance of the non-conservative positional forces described by the skew-symmetric matrix $\nu\mathbf{N}$. Thus, the Crandall gyropendulum can be treated as a conservative gyroscopic system perturbed by weak damping and non-conservative positional forces.

For $\sigma = \rho = 0$ the pendulum is stabilized by gyroscopic forces for $\Omega^2 > \Omega_0^{+2}$. At the points of the stability boundary $\Omega = \pm\Omega_0^+$ the spectrum of the gyropendulum has a pair of double purely imaginary eigenvalues $\pm i\omega_0$, where according to (7) and (8)

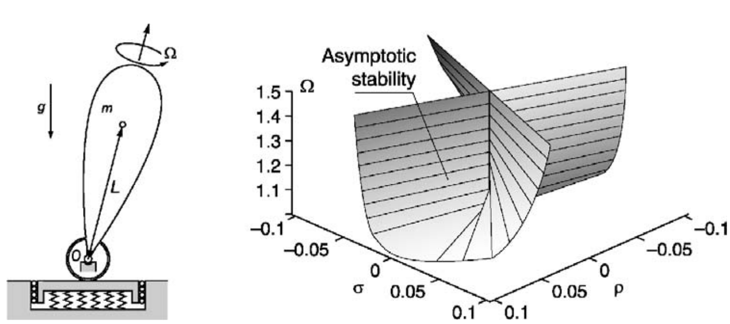

Fig. 3. Gyroscopic stabilization domain of the Crandall gyropendulum is a half of the Whitney umbrella.

$\Omega_0^+ = 2\alpha/\eta$ and $\omega_0 = \alpha$. Writing the Liénard–Chipart conditions for the characteristic polynomial of the Crandall gyropendulum with the damping forces we find the inequalities defining the asymptotic stability domain

$$\sigma + \rho > 0, \tag{25}$$

$$\eta^2\Omega^2 + (\sigma+\rho)^2 - 2\alpha^2 > 0, \tag{26}$$

$$\Omega^2 - \frac{(\sigma+\rho)^2\alpha^2}{\sigma\eta\rho + \eta\rho^2 - \rho^2} > 0. \tag{27}$$

Since the inequality (27) implies

$$\Omega^2 > \Omega_0^{+2} + \frac{1}{\rho}\frac{\alpha^2}{\eta^2}\frac{(\sigma\eta + \rho(\eta-2))^2}{\sigma\eta + \rho(\eta-1)} \geqslant \Omega_0^{+2}, \tag{28}$$

the asymptotic stability domain is given only by the conditions (25) and (27), which can be written in the form

$$\Omega > \Omega_{\mathrm{cr}}^+(\rho,\sigma), \qquad \Omega < \Omega_{\mathrm{cr}}^-(\rho,\sigma), \quad \sigma + \rho > 0, \tag{29}$$

where the critical values of the spin Ω as a function of the two damping parameters are

$$\Omega_{\mathrm{cr}}^{\pm}(\rho,\sigma) = \pm\frac{(\sigma+\rho)\alpha}{\sqrt{-\rho^2 + \rho^2\eta + \rho\eta\sigma}}. \tag{30}$$

Eqs. (30) describe two surfaces in the space of the parameters ρ, σ, and Ω. Both surfaces have Whitney umbrella singularities at the points $(0, 0, \pm\Omega_0^+)$. The surface $\Omega_{\mathrm{cr}}^+(\rho,\sigma)$ is shown in Fig. 3 for $\alpha = 1$ and $\eta = 2$. The inequality (25) selects the stable pocket of the Whitney umbrella. In spite of the fact that the formulae for the critical spin analogous to (30) were found by Crandall with the use of a perturbation technique, the singular nature of the asymptotic stability domain was not recognized in [8].

As it follows from the expressions (28), $\Omega_{\mathrm{cr}}^+ \geqslant \Omega_0^+$ and $\Omega_{\mathrm{cr}}^- \leqslant -\Omega_0^+$, which can be interpreted as the *destabilization* of the conservative gyroscopic system by the damping and non-conservative positional forces. The critical loads coincide only for the specific ratios of the coefficients of the stationary and rotating damping

$$\frac{b_s}{b_r} = \frac{\sigma}{\rho} = \frac{2-\eta}{\eta} = \frac{\Omega_0^+}{\omega_0} - 1 \tag{31}$$

in agreement with the result obtained in [8].

Conclusions

For a general linear mechanical system with two degrees of freedom the effect of weak damping and non-conservative positional forces on the gyroscopic stabilization has been studied. It was found that the boundary of the gyroscopic stabilization domain of the non-conservative system possesses Whitney umbrella singularity. Explicit analytical approximations of the boundary near the singularity were derived. The singularity is responsible for the high sensitivity of the critical gyroscopic parameter to small variations of the matrices of damping and circulatory forces. The price for the gyroscopic stabilization of a non-conservative system is generally higher values of the gyroscopic parameter and non-trivial choice of balance of damping and non-conservative positional forces. Finally, it was established that the stability boundary of the Crandall gyropendulum, considered as a mechanical example, consists of two pockets of two Whitney umbrellas.

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