Doklady Mathematics, Vol. 71, No. 3, 2005, pp. 470–475. Translated from Doklady Akademii Nauk, Vol. 402, No. 4, 2005, pp. 460–466.

MATHEMATICAL PHYSICS

Instability of Distributed Nonconservative Systems Caused by Weak Dissipation

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Presented by Academician V.A. Il'in October 19, 2004

Received October 22, 2004

We study the influence of weak dissipation on the stability of distributed nonconservative systems. A theory is constructed that qualitatively and quantitatively describes the destabilization paradox in such systems, i.e., a jump fall of the critical load and the critical frequency when small dissipative forces are taken into account [1–4]. This theory is based on an analysis of bifurcations of multiple eigenvalues of parameter-dependent non-self-adjoint differential operators. New explicit formulas are derived that describe the collapse of multiple eigenvalues with Keldysh chains of an arbitrary length for linear differential operators. It is shown that the destabilization paradox is related to a perturbation (caused by weak dissipation) in a double eigenvalue of a circulatory system with a Keldysh chain of length 2. Formulas are derived that describe the behavior of the eigenvalues of a nonconservative system with varying load and dissipation parameters. Explicit expressions for the jumps in the critical load and frequency corresponding to stability loss are found. The asymptotic stability domain in the parameter space of the system is approximated. The effect of stabilization (i.e., an increase in the critical load) of a distributed circulatory system by small dissipative forces is found, and the stabilization conditions are derived. As a mechanical example, we study the problem of stability of a viscoelastic column with small external and internal damping under the action of a follower force. An analytical formula for the critical load as a function of external and internal damping is derived.

1. Consider a generalized non-self-adjoint eigenvalue problem for a linear differential operator with boundary conditions [5–7]. Let L denote a linear differential operator of order m with respect to x. Its action on a smooth function $u(x)$ is defined by the relation

$$Lu = \sum_{j=0}^{m} l_j \frac{d^{m-j}u}{dx^{m-j}}. \qquad (1)$$

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The coefficients $l_j(x, \lambda, \mathbf{p})$ of L are smooth functions of x, and the function $l_0(x)$ on the interval $x \in [0, 1]$ is bounded from below by a positive constant. Moreover, it is assumed that the coefficients $l_j(x, \lambda, \mathbf{p})$ are analytic functions of the complex-valued spectral parameter λ and are smooth functions of the vector of real parameters $\mathbf{p} \in \mathbf{R}^n$.

The boundary condition matrix is a matrix $\mathbf{U} = [\mathbf{A}\ \mathbf{B}]$ of size $m \times 2m$ and rank m consisting of $m \times m$ blocks $\mathbf{A}$ and $\mathbf{B}$. Define a vector $\mathbf{u} = (\mathbf{u}(0), \mathbf{u}(1))$ of dimension $2m$, where $\mathbf{u}(0) = (u(0), u'_x(0), \dots, u_x^{(m-1)}(0))$ and $\mathbf{u}(1) = (u(1), u'_x(1), \dots, u_x^{(m-1)}(1))$ are formed from the values of $u(x)$ and its derivatives at the boundary points $x = 0$ and $x = 1$. Then,

$$\mathbf{U}\mathbf{u} = \mathbf{A}\mathbf{u}(0) + \mathbf{B}\mathbf{u}(1). \qquad (2)$$

It is assumed that the elements of $\mathbf{A}(\lambda, \mathbf{p})$ and $\mathbf{B}(\lambda, \mathbf{p})$ are analytic functions of λ and are smooth functions of $\mathbf{p} \in \mathbf{R}^n$.

On the interval $x \in [0, 1]$, we consider the eigenvalue problem for the differential operator L with the boundary conditions defined by $\mathbf{U}$:

$$L(x, \lambda, \mathbf{p})u = 0, \quad \mathbf{U}(\lambda, \mathbf{p})\mathbf{u} = 0. \qquad (3)$$

If $y_1(x), y_2(x), \dots, y_m(x)$ form a fundamental system of solutions to Eq. (3), then its general solution is given by $u(x) = \sum_{j=1}^{m} c_j y_j(x)$. A nontrivial solution to boundary value problem (3) exists if and only if the characteristic determinant vanishes:

$$\det(\mathbf{A}\mathbf{Y}(0) + \mathbf{B}\mathbf{Y}(1)) = 0, \qquad (4)$$

where the elements of $\mathbf{Y}(x)$ are defined by the relations $Y_{ij}(x) = y_{j_x}^{(i-1)}(x)$ for $i, j = 1, 2, \dots, m$. For a certain fixed vector $\mathbf{p} = \mathbf{p}_0$, the spectral parameter λ_0 for which problem (3) has a nontrivial solution u_0 is called an eigenvalue and the function u_0 is called the eigenfunction corresponding to λ_0. The eigenvalues of problem (3) are the roots of Eq. (4).

Consider an $m \times 2m$ matrix $\tilde{\mathbf{U}} = [\tilde{\mathbf{A}} \;\; \tilde{\mathbf{B}}]$, where $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are $m \times m$ matrices that generally depend on λ and $\mathbf{p}$. The matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are chosen so that a $2m \times 2m$ block matrix composed of $\mathbf{U}$ and $\tilde{\mathbf{U}}$ is nonsingular in the neighborhood of $\mathbf{p} = \mathbf{p}_0$ and $\lambda = \lambda_0$. The $m \times 2m$ matrices $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are defined as

$$\begin{bmatrix} -\tilde{\mathbf{V}} \\ \mathbf{V} \end{bmatrix}^* = \begin{bmatrix} -\mathbf{L}(0) & \mathbf{0} \\ \mathbf{0} & \mathbf{L}(1) \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \end{bmatrix}^{-1}, \tag{5}$$

where $\mathbf{0}$ is the $m \times m$ zero matrix and the asterisk denotes the Hermite conjugate. The matrices $\mathbf{L}(0)$ and $\mathbf{L}(1)$ in (5) are the values, at $x = 0$ and $x = 1$, of an $m \times m$ matrix $\mathbf{L}(x)$ whose elements $L_{ij}(x)$ are expressed in terms of the coefficients l_j of L and their derivatives with respect to x:

$$L_{ij}(x) = \sum_{k=i-1}^{m-j} (-1)^k C_k^{i-1} \frac{d^{k-i+1}}{dx^{k-i+1}} l_{m-j-k},$$

$$C_k^{i-1} = \begin{cases} \dfrac{k!}{(i-1)!(k-i+1)!}, & k \ge i-1 \ge 0 \\ 0, & k < i-1. \end{cases} \tag{6}$$

The adjoint L^* of L is defined as

$$L^* v = \sum_{j=0}^{m} (-1)^{m-j} \frac{d^{m-j}}{dx^{m-j}} (\overline{l_j(x)} v) \tag{7}$$

(see [5–7]). The eigenvalue problem for L^* with the boundary conditions

$$L^*(\bar{\lambda}, \mathbf{p}) v = 0, \quad \mathbf{V}(\bar{\lambda}, \mathbf{p}) \mathbf{v} = 0, \tag{8}$$

where $\mathbf{v} = (\mathbf{v}(0), \mathbf{v}(1))$ and the vectors $\mathbf{v}(0) = (v(0), v'_x(0), \dots, v_x^{(m-1)}(0))$ and $\mathbf{v}(1) = (v(1), v'_x(1), \dots, v_x^{(m-1)}(1))$ are formed from the values of $\mathbf{v}(x)$ and its derivatives at $x = 0$ and $x = 1$, is called the problem adjoint to (3).

Assume that, at the point $\mathbf{p}_0$ and its neighborhood, problem (3) has a discrete spectrum that contains a μ-multiple eigenvalue λ_0, with the Keldysh chain of length μ consisting of a single eigenfunction u_0 and the associated functions $u_1, u_2, \dots, u_{\mu-1}$. Let $L_0 = L(\lambda_0, \mathbf{p}_0)$ and $\mathbf{U}_0 = \mathbf{U}(\lambda_0, \mathbf{p}_0)$. The functions forming the Keldysh chain satisfy equations with the boundary conditions [5–7]

$$L_0 u_0 = 0, \quad \mathbf{U}_0 \mathbf{u}_0 = 0;$$
$$L_0 u_j = -\sum_{r=1}^{j} \frac{1}{r!} L_\lambda^{(r)} u_{j-r}, \quad \mathbf{U}_0 \mathbf{u}_j = -\sum_{r=1}^{j} \frac{1}{r!} \mathbf{U}_\lambda^{(r)} \mathbf{u}_{j-r}, \tag{9}$$
$$j = 1, 2, \dots, \mu-1,$$

where the partial derivatives are calculated at $\lambda = \lambda_0$ and $\mathbf{p} = \mathbf{p}_0$. The Keldysh chain for the complex conjugate eigenvalue $\bar{\lambda}_0$ of the Hermitian conjugate L_0^* of L_0 satisfies the equations

$$L_0^* v_0 = 0, \quad \mathbf{V}_0 \mathbf{v}_0 = 0;$$
$$L_0^* v_j = -\sum_{r=1}^{j} \frac{1}{r!} L_{\bar{\lambda}}^{*(r)} v_{j-r}, \quad \mathbf{V}_0 \mathbf{v}_j = -\sum_{r=1}^{j} \frac{1}{r!} \mathbf{V}_{\bar{\lambda}}^{(r)} \mathbf{v}_{j-r}, \tag{10}$$
$$j = 1, 2, \dots, \mu-1.$$

The functions of the Keldysh chains satisfy the orthogonality conditions

$$\sum_{r=1}^{j} \frac{1}{r!} ((L_\lambda^{(r)} u_{j-r}, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_\lambda^{(r)} \mathbf{u}_{j-r}) = 0, \tag{11}$$
$$j = 1, 2, \dots, \mu-1,$$

where $(u, v) = \int_0^1 u(x)\bar{v}(x)dx$ is the Hermitian scalar product of u and v.

Consider a smooth curve in an n-dimensional parameter space that depends on a real parameter $\varepsilon \ge 0$:

$$\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon \dot{\mathbf{p}} + \frac{\varepsilon^2}{2} \ddot{\mathbf{p}} + o(\varepsilon^2), \tag{12}$$

where the dotted letters denote the derivatives with respect to ε, which are calculated at $\varepsilon = 0$. Under perturbations of the parameters of form (12), the eigenvalue λ_0 with a Keldysh chain of length μ takes an increment that can be represented by the Newton–Puiseux series [8]

$$\lambda = \lambda_0 + \lambda_1 \varepsilon^{\frac{1}{\mu}} + \lambda_2 \varepsilon^{\frac{2}{\mu}} + \dots + \lambda_{\mu-1} \varepsilon^{\frac{(\mu-1)}{\mu}} + \lambda_\mu \varepsilon + \dots. \tag{13}$$

The coefficient λ_1 is found from the equations of the perturbation method and has the form

$$\lambda_1^\mu = -\frac{(L_1 u_0, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0}{\sum_{r=1}^{\mu} \frac{1}{r!} ((L_\lambda^{(r)} u_{\mu-r}, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_\lambda^{(r)} \mathbf{u}_{\mu-r})},$$
$$L_1 = \sum_{j=1}^{n} \frac{\partial L}{\partial p_j} \dot{p}_j, \quad \mathbf{U}_1 = \sum_{j=1}^{n} \frac{\partial \mathbf{U}}{\partial p_j} \dot{p}_j. \tag{14}$$

According to (13) and (14), under perturbation (12), the eigenvalue λ_0 splits into μ simple complex eigenvalues if the numerator of the expression for λ_1 does not vanish. The case $(L_1 u_0, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0 = 0$ is degenerate and requires separate consideration. For example, under perturbation (12) and the condition $\lambda_1 = 0$, the double ($\mu = 2$) eigenvalue with a Keldysh chain of length 2 splits according to the formula $\lambda = \lambda_0 + \varepsilon\lambda_2 + o(\lambda)$, where the coefficient λ_2 is determined from the quadratic equation

$$\lambda_2^2 + \lambda_2 \frac{(L_1u_0, v_1) + (L_1u_1, v_0) + (L'_{1\lambda}u_0, v_0) + \mathbf{v}_1^*\tilde{\mathbf{V}}_0^*\mathbf{U}_1\mathbf{u}_0 + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_1\mathbf{u}_1 + \mathbf{v}_0^*(\tilde{\mathbf{V}}^*\mathbf{U}_1)'_\lambda\mathbf{u}_0}{\sum_{r=1}^{2}\frac{1}{r!}((L_\lambda^{(r)}u_{2-r}, v_0) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_\lambda^{(r)}\mathbf{u}_{2-r})}$$
$$+ \frac{(L_2u_0, v_0) + (L_1\hat{w}_2, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*(\mathbf{U}_2\mathbf{u}_0 + \mathbf{U}_1\hat{\mathbf{w}}_2)}{\sum_{r=1}^{2}\frac{1}{r!}((L_\lambda^{(r)}u_{2-r}, v_0) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_\lambda^{(r)}\mathbf{u}_{2-r})} = 0. \quad (15)$$

The operator L_2 and the matrix $\mathbf{U}_2$ in Eq. (15) are given by

$$L_2 = \frac{1}{2}\sum_{j=1}^{n}\frac{\partial L}{\partial p_j}\ddot{p}_j + \frac{1}{2}\sum_{j,t=1}^{n}\frac{\partial^2 L}{\partial p_j \partial p_t}\dot{p}_j\dot{p}_t,$$
$$\mathbf{U}_2 = \frac{1}{2}\sum_{j=1}^{n}\frac{\partial \mathbf{U}}{\partial p_j}\ddot{p}_j + \frac{1}{2}\sum_{j,t=1}^{n}\frac{\partial^2 \mathbf{U}}{\partial p_j \partial p_t}\dot{p}_j\dot{p}_t, \quad (16)$$

and the function $\hat{w}_2$ solves the boundary value problem

$$L_0\hat{w}_2 = -L_1u_0, \quad \mathbf{U}_0\hat{\mathbf{w}}_2 = -\mathbf{U}_1\mathbf{u}_0, \quad (17)$$

where $\hat{\mathbf{w}}_2 = (\hat{w}_2(0), \hat{w}'_{2_x}(0), \ldots, \hat{w}_{2_x}^{(m-1)}(0), \hat{w}_2(1), \hat{w}'_{2_x}(1), \ldots, \hat{w}_{2_x}^{(m-1)}(1))$.

2. Now, we formulate the generalized eigenvalue problem arising in the stability analysis of the viscoelastic systems

$$L(\lambda, q, \mathbf{k})u \equiv N(q)u + \lambda D(\mathbf{k})u + \lambda^2 Mu = 0, \quad (18)$$

$$\mathbf{U}(q, \mathbf{k}, \lambda)\mathbf{u} \equiv \mathbf{U}_N(q)\mathbf{u} + \lambda\mathbf{U}_D(\mathbf{k})\mathbf{u} + \lambda^2\mathbf{U}_M\mathbf{u} = 0. \quad (19)$$

The coefficients of the differential operators N, D, and M of order m and the elements of the $m \times 2m$ matrices $\mathbf{U}_N$, $\mathbf{U}_D$, and $\mathbf{U}_M$ are assumed to be real. The operator N and the matrix $\mathbf{U}_N$ are smooth functions of the real load $q \geq 0$, and the elements of D (whose order does not exceed m) and $\mathbf{U}_D$ are smooth functions of the real dissipation parameter vector $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$; moreover, for $\mathbf{k} = 0$, we have $D(0) = 0$ and $\mathbf{U}_D(0) = \mathbf{0}$. It is also assumed that the operator M and the matrix $\mathbf{U}_M$ are parameter-independent. Thus, the perturbation of the system by small dissipative forces ($\|\mathbf{k}\| \ll 1$) is regular [8].

The unperturbed system

$$N(q)u + \lambda^2 Mu = 0, \quad \mathbf{U}_N(q)\mathbf{u} + \lambda^2\mathbf{U}_M\mathbf{u} = 0, \quad (20)$$

called circulatory, is assumed to have a discrete spectrum on the interval $0 \leq q < q_0$, which consists of simple purely imaginary eigenvalues $\lambda = i\omega$ and, hence, is stable. When $q = q_0$, it is assumed that there exists a pair of double eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$) with a Keldysh chain of length 2 (instability) [9]. All the remaining eigenvalues $\pm i\omega_{0,j}$ ($\omega_{0,j} > 0$) of the unperturbed system at $q = q_0$ are assumed to be simple and purely imaginary. Therefore, when dissipative forces are absent ($\mathbf{k} = 0$) and $q = q_0$, the nonconservative system is on the boundary between the domains of stability and instability (flutter).

The eigenfunction u_0 and the associated function u_1 corresponding to the eigenvalue $i\omega_0$ satisfy Eqs. (9), and the functions v_0 and v_1 satisfy Eqs. (10), in which we have to set $\mu = 2$. Since the elements of the operators and matrices involved in Eqs. (18) and (19) are real, we conclude that, when $\lambda = i\omega_0$, $\mathbf{k} = 0$, and $q = q_0$, the matrices $\mathbf{V}_0$ and $\tilde{\mathbf{V}}_0$ defined by (5) are real and the matrices $\frac{\partial \mathbf{V}}{\partial \bar{\lambda}}(\bar{\lambda}_0, \mathbf{p}_0)$ and $\frac{\partial \tilde{\mathbf{V}}}{\partial \bar{\lambda}}(\bar{\lambda}_0, \mathbf{p}_0)$ are purely imaginary. Since u_0 and v_0 are defined up to arbitrary multiplicative constants, and u_1 and v_1, up to the terms $\gamma_1 u_0$ and $\gamma_2 v_0$, respectively, where γ_1 and γ_2 are arbitrary constants, we choose real functions u_0 and v_0 and purely imaginary functions u_1 and v_1 that satisfy the orthogonality and normalization conditions

$$2i\omega_0(Mu_1, v_1) + (Mu_0, v_1) + (Mu_1, v_0)$$
$$+ (\tilde{\mathbf{V}}_0\mathbf{v}_1 + \tilde{\mathbf{V}}'_\lambda\mathbf{v}_0)^*(2i\omega_0\mathbf{U}_M\mathbf{u}_1 + \mathbf{U}_M\mathbf{u}_0)$$
$$+ \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_M\mathbf{u}_1 = 0,$$
$$2i\omega_0((Mu_1, v_0) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_M\mathbf{u}_1) = 1. \quad (21)$$

By using formulas (13)–(15) to analyze the collapse of the double eigenvalue $i\omega_0$ in problem (18), (19) under small perturbations in $\mathbf{k}$ and q, we find equations describing the motion of the eigenvalues over the complex plane (Fig. 1):

$$\left(\mathrm{Im}\lambda - \omega_0 + \mathrm{Re}\lambda + \frac{a}{2}\right)^2 - \left(\mathrm{Im}\lambda - \omega_0 - \mathrm{Re}\lambda - \frac{a}{2}\right)^2 = -2d, \quad (22)$$

$$\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^4 + \left(c - \frac{a^2}{4}\right)\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^2 = \frac{d^2}{4}, \quad (23)$$

$$(\mathrm{Im}\lambda - \omega_0)^4 - \left(c - \frac{a^2}{4}\right)(\mathrm{Im}\lambda - \omega_0)^2 = \frac{d^2}{4}, \quad (24)$$

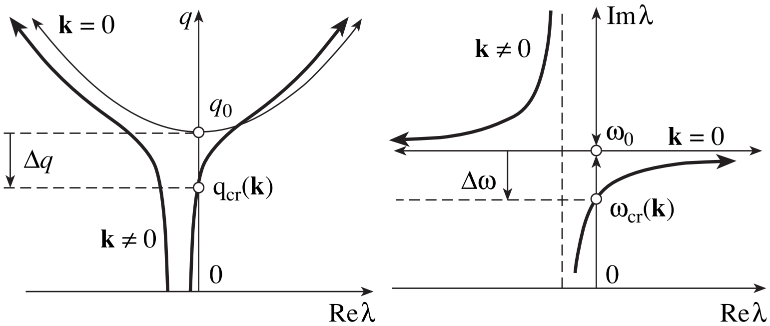

Fig. 1. Eigenvalue trajectories illustrating the destabilization paradox for $\mathbf{k} \neq 0$: the fall of the critical load and the critical frequency.

where

$$a = -\omega_0\langle \mathbf{h}, \mathbf{k}\rangle, \quad c = \tilde{f}(q - q_0) + \omega_0^2\langle \mathbf{Gk}, \mathbf{k}\rangle, \quad d = \omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{Hk}, \mathbf{k}\rangle), \tag{25}$$

the angular brackets denote the scalar product of real vectors in $\mathbf{R}^{n-1}$, the components of the real vector $\mathbf{f}$ and the real scalar $\tilde{f}$ have the form

$$\tilde{f} = \left(\frac{\partial N}{\partial q}u_0, v_0\right) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_N}{\partial q}\mathbf{u}_0, \quad f_r = \left(\frac{\partial D}{\partial k_r}u_0, v_0\right) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0, \tag{26}$$

and the components of the real vector $\mathbf{h}$ are defined as

$$ih_r = \left(\frac{\partial D}{\partial k_r}u_1, v_0\right) + \left(\frac{\partial D}{\partial k_r}u_0, v_1\right) + \mathbf{v}_1^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0 + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_1 + \mathbf{v}_0^*\left(\frac{\partial \tilde{\mathbf{V}}}{\partial \bar{\lambda}}\right)^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0. \tag{27}$$

In (25), the real matrix $\mathbf{H}$ has the components

$$H_{r\sigma} = \frac{1}{2}\left(\frac{\partial^2 D}{\partial k_r \partial k_\sigma}u_0, v_0\right) + \frac{1}{2}\mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial^2 \mathbf{U}_D}{\partial k_r \partial k_\sigma}\mathbf{u}_0, \quad r, \sigma = 1, 2, \ldots, n-1, \tag{28}$$

and the real matrix $\mathbf{G}$ is defined as

$$\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle = \sum_{r=1}^{n-1}\dot{k}_r\left(\left(\frac{\partial D}{\partial k_r}\hat{w}_2, v_0\right) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\hat{\mathbf{w}}_2\right), \tag{29}$$

where $\hat{w}_2$ is a solution to the boundary value problem

$$N(q_0)\hat{w}_2 - \omega_0^2 M\hat{w}_2 = \sum_{r=1}^{n-1}\dot{k}_r\frac{\partial D}{\partial k_r}u_0, \quad \mathbf{U}_N(q_0)\hat{\mathbf{w}}_2 - \omega_0^2\mathbf{U}_M\hat{\mathbf{w}}_2 = \sum_{r=1}^{n-1}\dot{k}_r\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0. \tag{30}$$

First, we consider the case of a circulatory system ($\mathbf{k} = 0$). Then, according to (25), $a = 0$, $c = \tilde{f}(q - q_0)$, $d = 0$, and Eqs. (23) and (24) yield the relations

$$q \le q_0: \mathrm{Re}\lambda = 0, \quad \mathrm{Im}\lambda = \omega_0 \pm \sqrt{\tilde{f}(q - q_0)}; \tag{31}$$

$$q \ge q_0: \mathrm{Re}\lambda = \pm\sqrt{-\tilde{f}(q - q_0)}, \quad \mathrm{Im}\lambda = \omega_0. \tag{32}$$

Equations (31) and (32) show that, in the case of $\tilde{f} < 0$, as q increases, the two simple purely imaginary eigenvalues move along the imaginary axis, collide at $q = q_0$, and then move away in a direction perpendicular to the imaginary axis to form a pair of simple complex eigenvalues (flutter). This behavior of eigenvalues is known as a strong interaction and is typical of circulatory systems.

For a fixed vector $\mathbf{k} \neq 0$ and $d \neq 0$, as q varies, the eigenvalues move over the complex plane along the branches of hyperbola (22). This hyperbola has the two asymptotes $\mathrm{Re}\lambda = -\frac{a}{2}$ and $\mathrm{Im}\lambda = \omega_0$, where a is defined by the first relation in (25). If $a > 0$, one of the two eigenvalues is on the left-hand side of the complex plane whereas the other crosses the imaginary axis and passes to the right-hand side at $q = q_{cr}(\mathbf{k})$. Thus, the condition $a > 0$ or, equivalently, $\langle \mathbf{h}, \mathbf{k}\rangle < 0$ is a necessary condition for asymptotic stability. The value of q at which one of the eigenvalues crosses the imaginary axis can be found from Eq. (23), where we set $\mathrm{Re}\lambda = 0$. This gives the relation $ca^2 = d^2$, which, in view of explicit expressions (25) for a, c, and d, becomes

$$q_{cr}(\mathbf{k}) = q_0 + \frac{(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{Hk}, \mathbf{k}\rangle)^2}{\tilde{f}\langle \mathbf{h}, \mathbf{k}\rangle^2} - \frac{\omega_0^2}{\tilde{f}}\langle \mathbf{Gk}, \mathbf{k}\rangle. \tag{33}$$

Thus, both eigenvalues are on the left-hand side of the complex plane if $q < q_{cr}(\mathbf{k})$ and $\langle \mathbf{h}, \mathbf{k}\rangle < 0$.

The asymptotic stability of system (18), (19) under perturbations of the parameters also depends on the behavior of the simple purely imaginary eigenvalues $\pm i\omega_{0,s}$ ($\omega_{0,s} > 0$), which move into the left-hand side of the complex plane if

$$\langle \mathbf{g}_s, \mathbf{k}\rangle > 0, \quad s = 1, 2, \ldots, \tag{34}$$

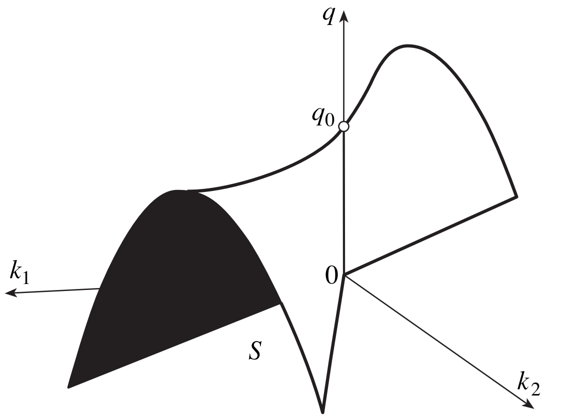

Fig. 2. Surface of $q_{cr}(k_1, k_2)$ given by Eq. (33) (Whitney umbrella).

where the components of the real vector $\mathbf{g}_s$ have the form

$$g_{s,r} = \left(\frac{\partial D}{\partial k_r} u_{0,s}, v_{0,s}\right) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \frac{\partial \mathbf{U}_D}{\partial k_r} \mathbf{u}_{0,s}, \quad r = 1, 2, \ldots, n-1. \tag{35}$$

The condition

$$\{\mathbf{k}: \langle \mathbf{f}, \mathbf{k} \rangle = 0, \langle \mathbf{h}, \mathbf{k} \rangle < 0\} \subset \{\mathbf{k}: \langle \mathbf{g}_s, \mathbf{k} \rangle > 0, s = 1, 2, \ldots\} \tag{36}$$

means that all the simple eigenvalues $\pm i\omega_{0,s}$ move into the left-hand side of the complex plane under small perturbations of q and $\mathbf{k}$. If (36) holds, the stability of system (18), (19) depends only on the collapse of the double eigenvalues $\pm i\omega_0$ and the surface $q_{cr}(k_1, k_2, \ldots, k_{n-1})$ approximated by Eq. (33) is the boundary of the asymptotic stability domain in a small neighborhood of the point $\mathbf{p}_0 = (0, \ldots, 0, q_0)$. When the dissipation parameter vector $\mathbf{k}$ consists of only two components k_1 and k_2, the surface $q_{cr}(k_1, k_2)$ given by (33) and bounding the asymptotic stability domain in the three-dimensional space of parameters k_1, k_2, q has a singularity (Whitney umbrella) at the point $(0, 0, q_0)$ (Fig. 2).

The level sets of function (33) are the boundaries of the stability domain in the space of parameters $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$. The level set $q_{cr} = q_0$, where q_0 is the critical value of q for the unperturbed circulatory system, is given by the equation

$$\langle \mathbf{f}, \mathbf{k} \rangle = \pm\omega_0 \langle \mathbf{h}, \mathbf{k} \rangle \sqrt{\langle \mathbf{G}\mathbf{k}, \mathbf{k} \rangle} - \langle \mathbf{H}\mathbf{k}, \mathbf{k} \rangle. \tag{37}$$

Real solutions to this equation exist if $\langle \mathbf{G}\mathbf{k}, \mathbf{k} \rangle \geq 0$. In this case, set (37) encloses the range of the dissipation parameter vector in $q_{cr}(\mathbf{k}) > q_0$, which means that the nonconservative system is stabilized by weak dissipation.

It follows from Eq. (33) that $q_{cr}(\mathbf{k})$ has a singularity at $\mathbf{k} = 0$ and the critical load, regarded as a function of $n - 1$ variables, has no limit as $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$ tends to zero. Nevertheless, the homogeneity of the numerator and the denominator of the rational part of $q_{cr}(\mathbf{k})$ guarantees the existence of $\lim_{\varepsilon \to 0} q_{cr}(\varepsilon\tilde{\mathbf{k}})$ for any direction $\tilde{\mathbf{k}}$ such that $\langle \mathbf{h}, \tilde{\mathbf{k}} \rangle \neq 0$. Substituting $\mathbf{k} = \varepsilon\tilde{\mathbf{k}}$ into Eq. (33), we find an explicit expression approximating the jump in the critical load due to small dissipative forces:

$$\Delta q \equiv q_0 - \lim_{\varepsilon \to 0} q_{cr}(\varepsilon\tilde{\mathbf{k}}) = -\frac{1}{\tilde{f}} \frac{\langle \mathbf{f}, \tilde{\mathbf{k}} \rangle^2}{\langle \mathbf{h}, \tilde{\mathbf{k}} \rangle^2}. \tag{38}$$

Substituting $\mathrm{Re}\lambda = 0$ into Eq. (22) yields an expression for the jump in the critical frequency due to weak dissipation $\mathbf{k} = \varepsilon\tilde{\mathbf{k}}$:

$$\Delta\omega \equiv \omega_0 - \lim_{\varepsilon \to 0} \omega_{cr}(\varepsilon\tilde{\mathbf{k}}) = -\frac{\langle \mathbf{f}, \tilde{\mathbf{k}} \rangle}{\langle \mathbf{h}, \tilde{\mathbf{k}} \rangle}. \tag{39}$$

If $\langle \mathbf{f}, \tilde{\mathbf{k}} \rangle = 0$, then there are no jumps in the critical load or the critical frequency ($\Delta q = 0$, $\Delta\omega = 0$).

3. Consider transverse vibrations (in a viscous medium) of a cantilevered column made of a Kelvin–Voigt viscoelastic material and loaded by a tangential follower force q at the free end (Fig. 3). In dimensionless variables, the stability problem is reduced to the study of the eigenvalue problem

$$Lu \equiv (1 + \lambda\eta)u''''_{xxxx} + qu''_{xx} + (\lambda^2 + \lambda\mu)u = 0, \tag{40}$$

$$u(0) = 0, \quad u'_x(0) = 0, \quad u''_{xx}(1) = 0, \quad u'''_{xxx}(1) = 0, \tag{41}$$

where $\mu \geq 0$ and $\eta \geq 0$ are the external and internal damping coefficients, respectively [2].

In the absence of damping ($\eta = \mu = 0$), the elastic column is stable for follower forces ranging in the interval $0 \leq q < q_0$, where $q_0 = 20.0509$ [1, 2]. When $q = q_0$, the spectrum of problem (40), (41) is discrete and consists of the pair of double eigenvalues $\pm i\omega_0$ ($\omega_0 = 11.0155$) and the simple eigenvalues $\pm i\omega_{0,s}$ ($s = 1, 2, \ldots$). The sequence of simple frequencies has the form

$$\omega_{0,1} = 53.7072, \quad \omega_{0,2} = 112.393, \quad \omega_{0,3} = 191.056, \ldots, \tag{42}$$

$$\omega_{0,s} = \pi^2 s^2 + O(s), \quad s \to \infty. \tag{43}$$

In the absence of damping, the eigenfunctions of problem (40), (41) were calculated in [1, 2]. The following asymptotic expansions are valid for eigenfunctions of high frequencies (43) [2]:

$$u_{0,s} = \sin(s\pi x) + O(s^{-1}), \quad v_{0,s} = \sin(s\pi x) + O(s^{-1}), \quad s \to \infty. \tag{44}$$

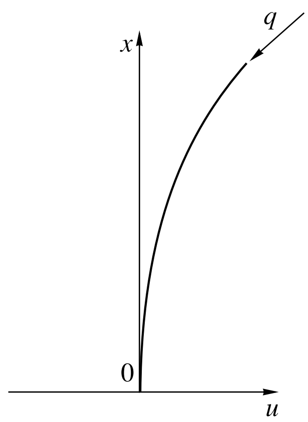

Fig. 3. Viscoelastic column loaded by a follower force.

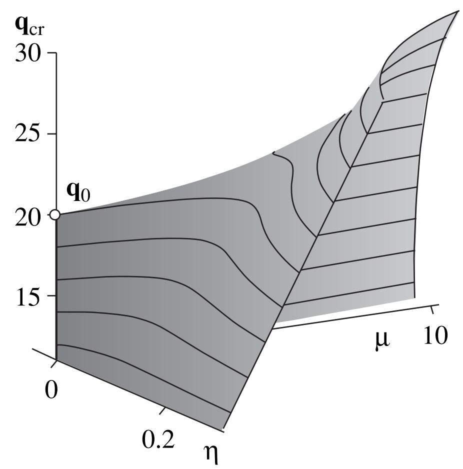

Fig. 4. Critical load as a function of the damping parameters.

Taking into account eigenfrequencies (42) and (43) and their eigenfunctions, we use formula (35) to find the vectors $\mathbf{g}_s$:

$$\mathbf{g}_1 = (35.4420, 0.00931), \quad \mathbf{g}_2 = (65.0334, 0.00445), \quad \mathbf{g}_3 = (104.536, 0.00261), \ldots, \tag{45}$$

$$\mathbf{g}_s = \frac{1}{2}(s^2\pi^2 + o(s^2), s^{-2}\pi^{-2} + o(s^{-2})), \quad s \to \infty. \tag{46}$$

Under conditions (34), where $\mathbf{k} = (\eta, \mu)$, all the simple eigenvalues move into the left half-plane under the action of weak dissipation.

Taking into account (26)–(30), which are calculated using the eigen- and associated functions of the double eigenvalue in problem (40), (41) without damping (as found in [9]), we use formula (33) to approximate the critical load as a function of the dissipation parameters:

$$q_{\rm cr}(\eta, \mu) = q_0 - \frac{1901.54\eta^2}{(14.3373\eta + 0.09078\mu)^2} + 12.6762\eta\mu + 0.05286\mu^2. \tag{47}$$

When $\mu = 0$ and $\eta \to 0$, formula (47) yields the critical load $q_{\rm cr} = 10.8 < q_0 = 20.05$, which agrees well with $q_{\rm cr} = 10.94$, found by numerical methods, in [1, 2]. Thus, the jump in the critical load as the damping parameters tend to zero is well approximated by formula (47). Combining the stability conditions obtained by analyzing the behavior of simple and double eigenvalues, we find that a viscoelastic column in a viscous medium is asymptotically stable in the neighborhood of the point $\eta = 0$, $\mu = 0$, $q = q_0$ if the following three conditions are satisfied:

$$\eta > 0, \quad \mu > -157.933\eta, \quad q < q_{\rm cr}(\eta, \mu). \tag{48}$$

Note that, for positive dissipation parameters, the first two conditions in (48) hold automatically. An approximation of asymptotic stability domain (48) in the space of η, μ, and q is shown in Fig. 4. It can be seen that the asymptotic stability domain has a singularity (Whitney umbrella) at the point $\eta = 0$, $\mu = 0$, and $q = q_0$; moreover, there exists a range of the damping parameters where $q_{\rm cr}(\eta, \mu) > q_0$. Thus, we can say that the nonconservative system is stabilized by low internal and external damping. Note that this effect has not been found in previous studies of distributed systems.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research jointly with the National Natural Science Foundation of China (project no. 02-01-39004), the Russian Foundation for Basic Research (project no. 03-01-00161), and CRDF-BRHE (Y1-MP-06-19).

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