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COLLAPSE OF THE KELDYSH CHAINS AND STABILITY OF CONTINUOUS NONCONSERVATIVE SYSTEMS*

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Abstract. In the present paper, eigenvalue problems for non-self-adjoint linear differential operators smoothly dependent on a vector of real parameters are considered. Bifurcation of eigenvalues along smooth curves in the parameter space is studied. The case of a multiple eigenvalue with the Keldysh chain of arbitrary length is investigated. Explicit expressions describing bifurcation of eigenvalues are found. The obtained formulas use eigenfunctions and associated functions of the adjoint eigenvalue problems as well as the derivatives of the differential operator taken at the initial point of the parameter space. These results are important for the stability problems and sensitivity analysis of nonconservative systems. As a mechanical application, the extended Beck problem of stability of an elastic column subjected to a partially tangential follower force is considered and discussed in detail.

Key words. nonconservative system, non-self-adjoint differential operator, Keldysh chain, multiple eigenvalue, bifurcation, stability boundary

AMS subject classifications. 34B08, 34D10, 34L16

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1. Introduction.

In nonconservative dynamic stability problems arising in mechanics and physics, energy is not conserved; it can be pumped into the system or taken out depending on problem parameters. Increase of energy leads to growth of vibrational amplitudes, i.e., to instability of vibrations. If energy is lost the amplitudes decay in time, which implies stability for the system. Unlike buckling problems of conservative systems for nonconservative ones, the dynamic method of stability study must be applied. Two types of instability in nonconservative systems are distinguished: static (divergence) and dynamic (flutter). Important examples of nonconservative systems are aircraft wings and panels vibrating in a flow, elastic missiles subjected to a jet thrust, which is a nonpotential follower force, and tubes conveying fluid; see Bolotin [1], Ziegler [2], Leipholz [3], and Paidoussis [4]. Sensitivity analysis of critical stability parameters for nonconservative systems was developed by Pedersen and Seyranian in [5]. A comprehensive review of nonconservative stability problems was given by Langthjem and Sugiyama in [6].

Non-self-adjoint operators naturally appear in nonconservative problems. In discrete problems such an operator is just a nonsymmetrical matrix. The general theory of non-self-adjoint operators going back to the works by Birkhoff was then developed by many mathematicians; see the review by Davies [7]. Keldysh [8] was the first to generalize the notion of the Jordan chain of vectors to a wide class of non-self-adjoint operators. For that reason it was called the Keldysh chain; see Gohberg and Krein [9] and Gohberg, Lancaster, and Rodman [10]. In the work by Vishik and Lyusternik [11], the perturbation theory for nonsymmetrical matrices and non-self-adjoint differential

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operators, $L = L_0 + \epsilon L_1$ with ϵ as a small parameter, was developed. This practical and constructive theory allows one to find the perturbation coefficients of eigenvalues and eigenvectors in an explicit form. The paper by Vishik and Lyusternik [11] was not widely known in the Western (and even Russian) literature on the subject for a long time. However, its importance was highly appreciated in the paper by Moro, Burke, and Overton [12]. We extend this perturbation theory to multiparameter bifurcation analysis of eigenvalues of non-self-adjoint differential operators.

It is known that in the generic case the spectrum of a multiparameter family of nonsymmetrical matrices contains multiple eigenvalues with the Jordan chains; see Arnold [13]. In many important cases multiple eigenvalues define geometrical properties of the stability boundary of a corresponding system. At the same time, multiple eigenvalues create considerable mathematical difficulties due to their nondifferentiability with respect to parameters. An effective tool for analysis of stability boundary is the study of bifurcations of multiple eigenvalues due to change of parameters. For the discrete case, this method based on the perturbation theory [11] was developed in the works by Seyranian [14], Mailybaev and Seyranian [15], and Seyranian and Kirillov [16]. To perform the stability analysis in the continuous case, we need to consider bifurcations of eigenvalues in multiparameter families of non-self-adjoint differential operators. The study of generic properties of the spectrum of the multiparameter family of non-self-adjoint differential operators remains a difficult problem. It seems that in the infinite-dimensional case there is still no analogue to the Arnold theory of versal deformations of matrices, allowing us to classify the generic singularities of the bifurcation and stability diagrams: even in the self-adjoint case the progress is quite slow; see Teytel [17].

In our paper we combine the ideas of [8], [11], and [14]. This allows us to find explicit formulas describing bifurcation of multiple eigenvalues with the Keldysh chain of any length. These formulas suit for a wide class of non-self-adjoint eigenvalue problems arising in applications and take into account parameters both in the differential expression and in the boundary conditions. Besides, our approach allows one to study bifurcations of multiple eigenvalues both in regular and degenerate cases. An analogous approach was applied by Seyranian and Kliem to the investigation of stability problems for continuous conservative systems with gyroscopic forces [18].

The paper is organized in the following way. In section 2 basic relations for eigenvalue problems with general linear differential operators and boundary conditions are introduced.

In section 3 it is supposed that the differential expression and boundary conditions smoothly depend on a vector of real parameters. A formula describing splitting of a multiple eigenvalue with the Keldysh chain of arbitrary length depending on a change of the parameters is derived. Both regular and degenerate cases are studied. Finally, a formula for bifurcation of a semisimple multiple eigenvalue is obtained. These formulas generalize the results for splitting of eigenvalues obtained earlier for the finite-dimensional case in [14], [15], and [16]. The obtained formulas take into account both variations of the differential expression and boundary conditions due to change of the parameters and can be applied to a wide class of nonconservative problems.

In section 4 we apply the results of previous sections to stability problems of general nonconservative (circulatory) systems. General solutions in an explicit form via eigenvalues and eigenvectors are obtained. Then stability boundaries in the parameter space are investigated. It is shown that the smooth parts of the boundaries correspond to either simple zero (stability-divergence boundary) or double positive eigenvalues with the Keldysh chain of length 2 (stability-flutter boundary). Normal vectors to the boundaries between stability, flutter, and divergence domains are found. It is remarkable that only information at a point of the stability boundary is needed for the calculation of the vectors.

Section 5 treats a mechanical example—an elastic column loaded by the partially tangential follower force. This problem is referred to as the extended Beck column. The stability boundaries of this two-parameter continuous system are carefully investigated and analyzed. Explicit expressions for eigenfunctions and associated functions are derived. With the use of these expressions, the linear and quadratic approximations of the stability boundary at its regular and singular points are constructed. The behavior of eigenvalues in the vicinity of the stability boundaries is studied by the perturbation approach, showing a good agreement with the numerical results.

2. Basic relations. Using the notation of Naimark [19] we consider an eigenvalue problem for a linear differential operator L defined by

$$l(u) = \lambda u, \quad U^s(u) = 0, \quad s = 1, \dots, m, \tag{2.1}$$

$$l(u) \equiv \sum_{i=0}^{m} a_i \frac{d^{m-i}u}{dx^{m-i}}, \quad U^s(u) \equiv \sum_{i=0}^{m-1} \left(\alpha_i^s \frac{d^i u}{dx^i}\bigg|_{x=0} + \beta_i^s \frac{d^i u}{dx^i}\bigg|_{x=1} \right).$$

The operators $U^s(u)$ are linear forms with respect to the variables $u(0),\ u'(0), \dots,$ $u^{(m-1)}(0);\ u(1),\ u'(1), \dots,\ u^{(m-1)}(1)$. These variables are values of the function $u \in C^{(m)}[0,1]$ and its derivatives up to $(m-1)$th order evaluated at the points $x = 0$ and $x = 1$. It is assumed that the forms U^s, $s = 1, 2, \dots, m$, are linearly independent.

The differential expression

$$l^*(v) \equiv \sum_{i=0}^{m} (-1)^{m-i}\ \overline{a_i}\ \frac{d^{m-i}v}{dx^{m-i}},$$

where the overbar denotes complex conjugation, is called *adjoint* to the differential expression $l(u)$ [19]. With the use of integration by parts it can be shown that

$$\int_0^1 l(u)\bar{v}dx = P(\alpha, \beta) + \int_0^1 u\overline{l^*(v)}dx, \tag{2.2}$$

where $P(\alpha, \beta)$ is a bilinear form of the variables

$$\alpha = (u(0), u'(0), \dots, u^{(m-1)}(0), u(1), u'(1), \dots, u^{(m-1)}(1)), \tag{2.3}$$

$$\beta = (v(0), v'(0), \dots, v^{(m-1)}(0), v(1), v'(1), \dots, v^{(m-1)}(1)). \tag{2.4}$$

Let us choose the forms $U^{m+1}, U^{m+2}, \dots, U^{2m}$ so that $U^1, U^2, \dots, U^{2m}$ are linearly independent. Then variables (2.3) can be expressed as linear combinations of the forms $U^1, U^2, \dots, U^{2m}$. Substituting these linear combinations into (2.2), we get the Lagrange identity [19]

$$(l(u), v) - (u, l^*(v)) = U^1 V^{2m} + \cdots + U^{2m} V^1, \tag{2.5}$$

where $(u, v) = \int_0^1 u(x)\overline{v}(x)dx$ is the inner product of the functions $u, v \in C^m[0,1]$.

The coefficients at $U^1, U^2, \ldots, U^{2m}$ are linear forms with respect to variables (2.4) and are denoted by $V^{2m}, \ldots, V^2, V^1$, respectively. The forms $V^1, V^2, \ldots, V^{2m}$ are linearly independent [19]. The boundary conditions

$$V^s(v) = 0, \quad s = 1, \ldots, m,$$

are called *adjoint* to boundary conditions (2.1). The differential operator L^*, corresponding to the differential expression $l^*(v)$ and to the adjoint boundary conditions, is referred to as adjoint to the operator L, and we say that the eigenvalue problem

$$l^*(v) = \overline{\lambda} v, \quad V^s(v) = 0, \quad s = 1, \ldots, m, \tag{2.6}$$

is adjoint to eigenvalue problem (2.1).

Due to the boundary conditions in (2.1) and (2.6), identity (2.5) for the adjoint operators L and L^* takes a simple form: $(l(u), v) = (u, l^*(v))$. If we consider differential expression $l(u)$ and assume that the function u satisfies the nonhomogeneous boundary conditions

$$U^s(u) = G^s, \quad s = 1, \ldots, m, \tag{2.7}$$

then the Lagrange identity (2.5) yields

$$(l(u), v) - (u, l^*(v)) = G^1 V^{2m} + \cdots + G^m V^{m+1}. \tag{2.8}$$

This is valid since v satisfies the boundary conditions in (2.6).

3. Collapse of the Keldysh chain. Suppose that in eigenvalue problem (2.1) the coefficients of the differential expression $l(u)$ and the coefficients of the forms $U^s(u)$ are real functions, *smoothly* dependent on a vector of real parameters $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ on an open set $\Omega \subset R^n$. Let λ_0 be an eigenvalue of the operator L at the point $\mathbf{p} = \mathbf{p}_0$. We are interested in bifurcation of eigenvalues along the curves $\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon \mathbf{e} + \epsilon^2 \mathbf{d} + o(\epsilon^2)$, emitted from the initial point $\mathbf{p}_0$ in the parameter space. The vector $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ defines a direction of the curve, and $\epsilon \geq 0$ is a small parameter.

Due to variation of parameters the differential expression $l(u)$ and the forms $U^s(u)$ are expanded as

$$l(u) = l_0(u) + \epsilon l_1(u) + \epsilon^2 l_2(u) + \cdots, \quad U^s(u) = U_0^s(u) + \epsilon U_1^s(u) + \epsilon^2 U_2^s(u) + \cdots, \tag{3.1}$$

where $l_0 = l(u)|_{\mathbf{p}=\mathbf{p}_0}$, $U_0^s = U^s(u)|_{\mathbf{p}=\mathbf{p}_0}$, the differential expressions $l_1(u)$, $l_2(u)$ look like

$$l_1(u) = \sum_{i=1}^{n} e_i \frac{\partial l}{\partial p_i}(u), \quad l_2(u) = \sum_{i=1}^{n} d_i \frac{\partial l}{\partial p_i}(u) + \frac{1}{2} \sum_{i,j=1}^{n} e_i e_j \frac{\partial^2 l}{\partial p_i \partial p_j}(u), \tag{3.2}$$

and for the forms $U_1^s(u)$, $U_2^s(u)$ we have

$$U_1^s(u) = \sum_{i=1}^{n} e_i \frac{\partial U^s}{\partial p_i}(u), \quad U_2^s(u) = \sum_{i=1}^{n} d_i \frac{\partial U^s}{\partial p_i}(u) + \frac{1}{2} \sum_{i,j=1}^{n} e_i e_j \frac{\partial^2 U^s}{\partial p_i \partial p_j}(u). \tag{3.3}$$

All the derivatives in formulas (3.2) and (3.3) are evaluated at the point $\mathbf{p} = \mathbf{p}_0$. Thus, we deal with the *regular* perturbations which do not increase the order of the nonperturbed operator $L_0 = L(\mathbf{p}_0)$ [11].

Consider an eigenvalue λ_0 with *the Keldysh chain of length* $k > 0$. This means that at $\mathbf{p} = \mathbf{p}_0$ there exist an eigenfunction $u_0(x)$ and *associated* functions $u_1(x)$, $u_2(x), \ldots, u_{k-1}(x)$, corresponding to the λ_0 and satisfying the equations and the boundary conditions

$$\begin{gathered} l_0(u_0) = \lambda_0 u_0, \qquad U_0^s(u_0) = 0; \\ l_0(u_i) = \lambda_0 u_i + u_{i-1}, \quad U_0^s(u_i) = 0; \\ i = 1, \ldots, k-1; \quad s = 1, \ldots, m. \end{gathered} \tag{3.4}$$

For the adjoint operator L^* we have

$$\begin{gathered} l_0^*(v_0) = \overline{\lambda_0} v_0, \qquad V_0^s(v_0) = 0; \\ l_0^*(v_i) = \overline{\lambda_0} v_i + v_{i-1}, \quad V_0^s(v_i) = 0; \\ i = 1, \ldots, k-1; \quad s = 1, \ldots, m. \end{gathered} \tag{3.5}$$

The notion of the Keldysh chain is an analogue of the Jordan chain of vectors when we consider eigenvalue problems for differential operators [8], [9], [19]. Eigenfunctions and associated functions of adjoint operators L and L^* are related by the following conditions:

$$(u_j, v_0) = 0, \quad j = 0, \ldots, k-2, \quad (u_{k-1}, v_0) \equiv (u_0, v_{k-1}) \neq 0; \tag{3.6}$$

$$(u_{j-1}, v_i) \equiv (u_j, v_{i-1}), \quad i, j = 1, \ldots, k-1. \tag{3.7}$$

This naturally follows from (3.4) and (3.5) with the relation $(l(u), v) = (u, l^*(v))$ stated for the adjoint operators.

A variation of the vector of parameters $\mathbf{p} = \mathbf{p}_0 + \epsilon \mathbf{e} + o(\epsilon)$ causes perturbation of eigenvalues and eigenfunctions. In the case of a multiple eigenvalue with the Keldysh chain of length k, the expansions for eigenvalues and eigenfunctions contain terms with fractional powers of the small parameter $\epsilon^{j/k}$, $j = 0, 1, 2, \ldots$ [11]:

$$\lambda = \lambda_0 + \epsilon^{1/k}\lambda_1 + \epsilon^{2/k}\lambda_2 + \cdots, \quad u = u_0 + \epsilon^{1/k} w_1 + \epsilon^{2/k} w_2 + \cdots. \tag{3.8}$$

Substituting expansions (3.1) and (3.8) into eigenvalue problem (2.1), we get expressions that determine the first order perturbations of the eigenvalue λ_0 and eigenfunction u_0,

$$l_0(w_j) - \lambda_0 w_j = \lambda_j u_0 + \sum_{i=1}^{j-1>0} \lambda_{j-i} w_i, \quad U_0^s(w_j) = 0, \quad j = 1 \ldots k-1, \tag{3.9}$$

$$l_0(w_k) - \lambda_0 w_k = \lambda_k u_0 - l_1(u_0) + \sum_{i=1}^{k-1>0} \lambda_{k-i} w_i, \quad U_0^s(w_k) = -U_1^s(u_0). \tag{3.10}$$

The functions w_j can be found from (3.4) and (3.9) in the form

$$w_j = \lambda_1^j u_j + \sum_{p=0}^{j-1} \gamma_{jp} u_p, \quad j = 1, \ldots, k-1, \tag{3.11}$$

where γ_{jp} are arbitrary constants.

Consider the inner product of the function v_0 with the left- and right-hand sides of (3.10). Using then expression (3.11) for w_j, equations (3.6) and (3.7), and the Lagrange identity (2.8), which in this case has the form

$$(l_0(w_k) - \lambda_0 w_k, v_0) - (w_k, l_0^*(v_0) - \overline{\lambda_0} v_0) = -\sum_{s=1}^{m} U_1^s(u_0) V_0^{2m-s+1}(v_0),$$

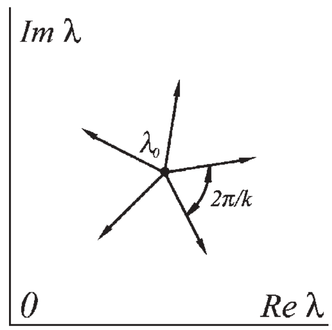

FIG. 3.1. *Splitting of the multiple eigenvalue* λ_0 *with the Keldysh chain of length* $k = 5$.

we get the coefficient λ_1 in the expansion of the eigenvalue λ,

$$\lambda_1^k = \frac{(l_1(u_0), v_0) - \sum_{s=1}^m U_1^s(u_0)V_0^{2m-s+1}(v_0)}{(u_{k-1}, v_0)}. \tag{3.12}$$

Introducing scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ of the vectors $\mathbf{a}, \mathbf{b} \in R^n$ and taking into account expressions (3.2) and (3.3) we can rewrite (3.12) in the form [14]

$$\lambda_1 = \sqrt[k]{\langle \mathbf{f}_k, \mathbf{e} \rangle + i\langle \mathbf{g}_k, \mathbf{e} \rangle}, \tag{3.13}$$

where the real vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ correspond to the k-fold eigenvalue λ_0 at the point $\mathbf{p} = \mathbf{p}_0$ and their components are

$$f_k^j + ig_k^j = \frac{(\frac{\partial l}{\partial p_j}(u_0), v_0) - \sum_{s=1}^m \frac{\partial U^s}{\partial p_j}(u_0)V_0^{2m-s+1}(v_0)}{(u_{k-1}, v_0)}, \quad j = 1, \dots, n. \tag{3.14}$$

The right-hand side of (3.13) takes k complex values. If the radicand in (3.13) is not zero, the expression $\lambda = \lambda_0 + \epsilon^{1/k}\lambda_1 + o(\epsilon^{1/k})$ describes the splitting of the k-fold eigenvalue with a change of parameters along a curve emitted in the direction $\mathbf{e}$, as shown in Figure 3.1.

We emphasize that for real eigenvalue λ_0 the vector $\mathbf{g}_k$=0 since the eigenfunctions and associated functions can be chosen real, and the coefficient λ_1 does not depend on normalization conditions.

After splitting, the length of the Keldysh chain decreases from k to 1, and we say that *collapse of the Keldysh chain* occurs.

In particular, (3.8) and (3.13) describe the behavior of a simple eigenvalue for $k = 1$, and for $k = 2$-splitting of a double eigenvalue λ_0 with the Keldysh chain of length 2, which are the most important cases in applications.

Consider now for a double eigenvalue the *degenerate* case

$$\langle \mathbf{f}_2, \mathbf{e}_* \rangle + i\langle \mathbf{g}_2, \mathbf{e}_* \rangle = 0. \tag{3.15}$$

It follows from condition (3.15) that the coefficient λ_1 in expansions (3.8) becomes zero. Substitution of expansions (3.8) into eigenvalue problem (2.1) gives equations determining second order terms λ_2 and w_2,

$$l_0(w_2) - \lambda_0 w_2 = \lambda_2 u_0 - l_1(u_0), \quad U_0^s(w_2) = -U_1^s(u_0), \tag{3.16}$$

$$(3.17)\quad l_0(w_4)-\lambda_0 w_4=\lambda_4 u_0-l_2(u_0)+\lambda_2 w_2-l_1(w_2),\quad U_0^s(w_4)=-U_1^s(w_2)-U_2^s(u_0).$$

Multiplying both parts of (3.17) by v_0 and using the Lagrange identity (2.8), we get

$$(3.18)\ \lambda_2(w_2,v_0)-(l_1(w_2),v_0)-(l_2(u_0),v_0)+\sum_{s=1}^{m}(U_1^s(w_2)+U_2^s(u_0))V_0^{2m-s+1}(v_0)=0.$$

Multiplication of (3.16) by v_1 with the use of (2.8) and (3.5) gives the term (w_2, v_0),

$$(3.19)\qquad (w_2,v_0)=\lambda_2(u_0,v_1)-(l_1(u_0),v_1)+\sum_{s=1}^{m}U_1^s(u_0)V_0^{2m-s+1}(v_1).$$

The solution of (3.16) has the form $w_2 = \lambda_2 u_1 + \hat{w}_2 + \gamma u_0$, where γ is an arbitrary constant and $\hat{w}_2$ is a particular solution of the boundary value problem

$$(3.20)\qquad l_0(\hat{w}_2)-\lambda_0\hat{w}_2=-l_1(u_0),\quad U_0^s(\hat{w}_2)=-U_1^s(u_0).$$

The solution of boundary value problem (3.20) exists due to degeneration condition (3.15), playing here the role of the solvability condition. Substituting (3.19) and the expression for w_2 into (3.18) we get the quadratic equation in λ_2,

$$(3.21)\qquad \lambda_2^2+\lambda_2 a_1+a_2=0.$$

The coefficients a_1 and a_2 are determined by the expressions

$$(3.22)\quad a_1=\frac{\sum_{s=1}^{m}[U_1^s(u_0)V_0^{2m-s+1}(v_1)+U_1^s(u_1)V_0^{2m-s+1}(v_0)]}{(u_0,v_1)}-\frac{(l_1(u_0),v_1)+(l_1(u_1),v_0)}{(u_0,v_1)},$$

$$(3.23)\quad a_2=\frac{-(l_1(\hat{w}_2),v_0)-(l_2(u_0),v_0)+\sum_{s=1}^{m}(U_1^s(\hat{w}_2)+U_2^s(u_0))V_0^{2m-s+1}(v_0)}{(u_0,v_1)}.$$

These coefficients can be written in the form

$$(3.24)\quad a_1=\langle\mathbf{h},\mathbf{e}_*\rangle+i\langle\mathbf{k},\mathbf{e}_*\rangle,\quad a_2=\langle\mathbf{H}\mathbf{e}_*,\mathbf{e}_*\rangle+i\langle\mathbf{K}\mathbf{e}_*,\mathbf{e}_*\rangle-\langle\mathbf{f}_2,\mathbf{d}\rangle-i\langle\mathbf{g}_2,\mathbf{d}\rangle,$$

where components of the real vectors $\mathbf{h}$, $\mathbf{k}$ are defined by the relationship

$$(3.25)\quad h^j+ik^j =\frac{\sum_{s=1}^{m}[\frac{\partial U^s}{\partial p_j}(u_0)V_0^{2m-s+1}(v_1)+\frac{\partial U^s}{\partial p_j}(u_1)V_0^{2m-s+1}(v_0)]}{(u_0,v_1)}-\frac{(\frac{\partial l}{\partial p_j}(u_0),v_1)+(\frac{\partial l}{\partial p_j}(u_1),v_0)}{(u_0,v_1)},$$

and the real symmetric matrices $\mathbf{H}$ and $\mathbf{K}$ can be found according to (3.23) and (3.24). Thus, bifurcation of the double eigenvalue λ_0 in degenerate case (3.15) is described by the formula $\lambda = \lambda_0 + \epsilon\lambda_2 + o(\epsilon)$, where λ_2 are the two roots of (3.21).

Finally, we consider at $\mathbf{p} = \mathbf{p}_0$ a so-called [18] semisimple eigenvalue λ_0 of multiplicity k with k linearly independent eigenfunctions $u_0^1, u_0^2, \dots, u_0^k$ satisfying eigenvalue problem (2.1). The complex-conjugate $\overline{\lambda}_0$ is the semisimple eigenvalue of adjoint eigenvalue problem (2.6) with the eigenfunctions $v_0^1, v_0^2, \dots, v_0^k$.

Expansion of the parameters $\mathbf{p} = \mathbf{p}_0 + \epsilon\mathbf{e} + o(\epsilon)$ causes perturbations of the eigenvalue and eigenfunctions, which can be expressed as Taylor series with respect to the small parameter ϵ [11],

$$(3.26)\qquad \lambda=\lambda_0+\epsilon\lambda_1+o(\epsilon),\quad u=w_0+\epsilon w_1+o(\epsilon).$$

Substituting these expansions as well as expansions (3.1) into eigenvalue problem (2.1) we get equations determining the functions w_0, w_1:

$$l_0(w_0) - \lambda_0 w_0 = 0, \quad U_0^s(w_0) = 0; \tag{3.27}$$

$$l_0(w_1) - \lambda_0 w_1 = -l_1(w_0) + \lambda_1 w_0, \quad U_0^s = -U_1^s(w_0). \tag{3.28}$$

A general solution of eigenvalue problem (3.27) has the form

$$w_0 = \sum_{i=1}^{k} \gamma_i u_0^i$$

with unknown coefficients γ_i. Taking the inner product of (3.28) and the functions $v_0^1, v_0^2, \dots, v_0^k$ and using the Lagrange identity

$$(l_0(w_0) - \lambda_0 w_0, v_0^j) - (w_0, l_0^*(v_0^j) - \overline{\lambda_0} v_0^j) = -\sum_{s=1}^{m} U_1^s(w_0) V_0^{2m-s+1}(v_0^j),$$

we come to the system of equations on the coefficients $\gamma_1, \gamma_2, \dots, \gamma_k$,

$$\sum_{i=1}^{k} \left((l_1(u_0^i), v_0^j) - \sum_{s=1}^{m} U_1^s(u_0^i) V_0^{2m-s+1}(v_0^j) - \lambda_1 (u_0^i, v_0^j) \right) \gamma_i = 0, \quad j = 1, \dots, k.$$

This system has a nontrivial solution if and only if

$$\det \left[(l_1(u_0^i), v_0^j) - \sum_{s=1}^{m} U_1^s(u_0^i) V_0^{2m-s+1}(v_0^j) - \lambda_1 (u_0^i, v_0^j) \right] = 0. \tag{3.29}$$

The coefficients of (3.29) can also be expressed in terms of the vector of variation $\mathbf{e}$. For the sake of convenience we suppose that the eigenfunctions satisfy the orthonormality conditions

$$(u_0^\sigma, v_0^j) = \delta_{\sigma j}, \quad \sigma, j = 1, \dots, k, \tag{3.30}$$

where $\delta_{\sigma j}$ is the Kronecker symbol. Introducing the real vectors $\mathbf{f}^{\sigma j}$ and $\mathbf{g}^{\sigma j}$ of dimension n with the components defined by the equation

$$f_r^{\sigma j} + i g_r^{\sigma j} = \left(\frac{\partial l}{\partial p_r}(u_0^\sigma), v_0^j \right) - \sum_{s=1}^{m} \frac{\partial U^s}{\partial p_r}(u_0^\sigma) V_0^{2m-s+1}(v_0^j), \quad r = 1, \dots, n, \tag{3.31}$$

we can write (3.29) in the following form:

$$\det[\langle \mathbf{f}^{\sigma j} + i\mathbf{g}^{\sigma j}, \mathbf{e} \rangle - \lambda_1 \delta_{\sigma j}] = 0, \quad \sigma, j = 1, \dots, k. \tag{3.32}$$

Equation (3.32) is a polynomial of kth order for the coefficients λ_1 in expansions (3.26), which describe splitting of a multiple eigenvalue λ_0.

4. Application to nonconservative stability problems. Let us consider nonconservative systems, described by a partial differential equation with the boundary conditions

$$\ddot{y} + l(y) = 0, \quad U^s(y) = 0, \quad s = 1, \dots, m, \tag{4.1}$$

where $y = y(x,t)$, dots mean differentiation with respect to time t, while $l(y)$ and $U^s(y)$ are, respectively, the linear differential expression in terms of $x \in [0,1]$ and the boundary forms defined in section 2. Such systems are usually called *circulatory systems* [1], [2], [3]. Note that damping and gyroscopic forces are not involved in such systems. However, circulatory forces play an important role in aeroelasticity, plasma physics, gyrodynamics, and other fields.

Looking up the solution of (4.1) in the form

$$y(x,t) = u(x)e^{\pm it\sqrt{\lambda}}$$

we come to the eigenvalue problem (2.1). Recall that the coefficients of the differential expression $l(u)$ and the coefficients of the forms $U^s(u)$ are real functions, smoothly dependent on a vector of real parameters $\mathbf{p} = (p_1, p_2, \dots, p_n)$. It follows from the basic theorems of the theory of ordinary differential equations [20] that solutions $z_1, \dots, z_m$ of (2.1) with the initial conditions (δ_{ij} is the Kronecker symbol)

$$z_i^{(j-1)}(0) = \delta_{ij}, \quad i,j = 1,\dots,m,$$

forming the fundamental system of solutions of (2.1), smoothly depend on λ and $\mathbf{p}$. The characteristic determinant $\Delta \equiv \det \left\| U^i(z_j) \right\|$ is thus a smooth function of the spectral parameter λ and the vector $\mathbf{p}$: $\Delta = \Delta(\lambda, \mathbf{p})$.

We assume that at some fixed value $\mathbf{p}_0$ of the vector $\mathbf{p}$ the spectrum of the operator L formed by the differential expression $l(u)$ and boundary conditions $U^s(u) = 0$ is discrete. The eigenvalues λ can be simple or multiple roots of the characteristic equation $\Delta(\lambda, \mathbf{p}_0) = 0$.

If all eigenvalues λ_j are simple, then a general solution of (4.1) has the form

$$y(x,t) = \sum_{j=1}^{\infty} u_0^j(x)(\alpha_j e^{it\sqrt{\lambda_j}} + \beta_j e^{-it\sqrt{\lambda_j}}) \tag{4.2}$$

with arbitrary constants α_j, β_j. This form is also valid for semisimple eigenvalues λ_0 of algebraic multiplicity k ($\lambda_j = \lambda_{j+1} = \dots = \lambda_{j+k-1} = \lambda_0$), which means that the number of linearly independent eigenfunctions μ corresponding to λ_0 is equal to k. However, when $\mu < k$ in the general solution of (4.1), the *secular* terms proportional to $t^\sigma e^{\pm it\sqrt{\lambda_0}}$ with $\sigma < k$ appear.

Let $\lambda = \lambda_0$ be the k-fold root of the characteristic equation $\Delta(\lambda, \mathbf{p}_0) = 0$, and let the functions $u_0(x)$, $u_1(x), \dots, u_{k-1}(x)$ satisfying (3.4) be the Keldysh chain of length k corresponding to λ_0. In this case $\mu = 1$, and the partial solution of the boundary value problem (4.1) has the form

$$y(x,t) = (\alpha e^{it\sqrt{\lambda_0}} + \beta e^{-it\sqrt{\lambda_0}}) \sum_{r=0}^{k-1} y_r(x) \frac{t^{k-r-1}}{(k-r-1)!}, \tag{4.3}$$

$$y_r(x) = \sum_{j=0}^{r} (-1)^{r-j} (2i\sqrt{\lambda_0})^{r-2j} C_{r-j}^{r-2j} u_{r-j}(x), \quad C_{r-j}^{r-2j} = \begin{cases} 0, & 2j > r, \\ \frac{(r-j)!}{j!(r-2j)!}, & 2j \le r, \end{cases}$$

which can be verified by direct substitution to (4.1).

In the more complicated case of the multiple eigenvalue λ_0 of multiplicity $\sum_{s=1}^{\mu} k_s$ with μ Keldysh chains of lengths $k_1, k_2, \dots, k_\mu$, respectively, the solution corresponding to λ_0 is a sum of functions (4.3) over all μ Keldysh chains,

$$y(x,t) = \sum_{s=1}^{\mu} (\alpha_s e^{it\sqrt{\lambda_0}} + \beta_s e^{-it\sqrt{\lambda_0}}) \sum_{r=0}^{k_s-1} y_r^s(x) \frac{t^{k_s-r-1}}{(k_s-r-1)!}, \tag{4.4}$$

where the functions $y_r^s(x)$ are related to the functions $u_r^s(x)$ by the second of equations (4.3).

From (4.2)–(4.4) it is obvious that the linear system described by (4.1) is stable if and only if all the eigenvalues λ are nonnegative and semisimple. If all λ are real and some of them negative, then the circulatory system is statically unstable (divergence). Existence of at least one complex eigenvalue or a multiple real positive eigenvalue with the Keldysh chain of length $k > 1$ means flutter instability. Multiple zero eigenvalue with the Keldysh chain of length $k > 1$ causes divergence.

Let us now study the stability boundaries of circulatory systems in the parameter space. If at the point $\mathbf{p} = \mathbf{p}_0$ the characteristic equation has the k-fold real root $\lambda = \lambda_0$, i.e., $\Delta(\lambda_0, \mathbf{p}_0) = \partial\Delta/\partial\lambda = \cdots = \partial^{k-1}\Delta/\partial\lambda^{k-1} = 0$, $\partial^k\Delta/\partial\lambda^k \neq 0$, then according to Malgrange's preparation theorem [21] there exists a neighborhood $U_0 \subset R \times R^n$ of the point $(\lambda_0, \mathbf{p}_0)$, where $\Delta(\lambda, \mathbf{p})$ has the form

$$\Delta(\lambda, \mathbf{p}) = \left[(\lambda - \lambda_0)^k + \sum_{i=0}^{k-1}(\lambda - \lambda_0)^i a_i(\mathbf{p})\right] b(\lambda, \mathbf{p}). \tag{4.5}$$

The functions $a_0(\mathbf{p}), \ldots, a_{k-1}(\mathbf{p})$ and $b(\lambda, \mathbf{p})$ are real and smooth, $a_i(\mathbf{p}_0) = 0$, and $b(\lambda_0, \mathbf{p}_0) \neq 0$.

Let, for example, λ_0 be a simple real root of the equation $\Delta(\lambda, \mathbf{p}_0) = 0$. Then due to (4.5) we can write $\lambda = \lambda_0 - a_0(\mathbf{p})$, and λ remains real and simple in some neighborhood of the point $\mathbf{p}_0$. We can conclude from this fact that if at $\mathbf{p} = \mathbf{p}_0$ all the eigenvalues of eigenvalue problem (2.1) are positive and simple, then $\mathbf{p}_0$ is the inner point of the stability domain of circulatory system (4.1).

Similarly, the points of the parameter space, corresponding to either simple zero eigenvalue or real double eigenvalue with the Keldysh chain of length 2, form smooth surfaces of dimension $n-1$. Indeed, if $\lambda_0 = 0$ at $\mathbf{p} = \mathbf{p}_0$, then in the vicinity of $\mathbf{p}_0$ we have $\lambda = -a_0(\mathbf{p})$. The equation $a_0(\mathbf{p}) = 0$ defines a hypersurface in the parameter space.

If λ_0 is a double eigenvalue, then according to (4.1) its behavior near the point $\mathbf{p}_0$ is described by the quadratic equation

$$(\lambda - \lambda_0)^2 + a_1(\mathbf{p})(\lambda - \lambda_0) + a_0(\mathbf{p}) = 0. \tag{4.6}$$

It follows from (4.6) that the eigenvalue $\lambda(\mathbf{p})$ remains double in the neighborhood of the point $\mathbf{p}_0$ if $\mathbf{p}$ belongs to the hypersurface $a_1^2(\mathbf{p}) - 4a_0(\mathbf{p}) = 0$.

It is clear that the stability of the system in a neighborhood of the point $\mathbf{p}_0$ belonging to these hypersurfaces depends on the behavior of the zero or the double eigenvalues due to change of parameters if all other eigenvalues at the point $\mathbf{p}_0$ are positive and semisimple. According to (3.8) and (3.13), where we should put $k = 1$ or $k = 2$, the behavior of the simple zero eigenvalue is described by the formula

$$\lambda = \epsilon\langle \mathbf{f}_1, \mathbf{e}\rangle + o(\epsilon), \tag{4.7}$$

and the splitting of the real double λ_0 is governed by the expression

$$\lambda = \lambda_0 \pm \sqrt{\epsilon\langle \mathbf{f}_2, \mathbf{e}\rangle} + o(\epsilon^{1/2}). \tag{4.8}$$

The inequality $\langle \mathbf{f}_1, \mathbf{e}\rangle > 0$ defines a set of directions $\mathbf{e}$ such that the curves $\mathbf{p} = \mathbf{p}(\epsilon)$ emitted along these vectors lie in the stability domain, i.e., *a tangent cone* to the stability domain. The eigenvalue λ becomes negative for $\langle \mathbf{f}_1, \mathbf{e}\rangle < 0$. Consequently, this inequality gives a tangent cone to the static instability (divergence) domain. The eigenvalue remains zero up to the terms of order ϵ^2 on the curves, emitted in the directions $\mathbf{e}$, such that $\langle \mathbf{f}_1, \mathbf{e}\rangle = 0$. Thus, the equation $\langle \mathbf{f}_1, \mathbf{p} - \mathbf{p}_0\rangle = 0$ defines a tangent plane to the surface, where the operator L has a simple zero eigenvalue. If other eigenvalues remain simple and positive on this surface, then it forms a boundary between stability and divergence domains. The vector $\mathbf{f}_1$ is the normal vector to the boundary and is directed to the stability domain.

Analyzing splitting of the double eigenvalue with the formula (4.8) we see that the points of the parameter space, corresponding to the real double eigenvalue with the Keldysh chain of length 2, belong to the smooth parts of the boundary between the flutter domain and the stability domain if $\lambda_0 > 0$ or the divergence domain if $\lambda_0 < 0$. In this case the vector $\mathbf{f}_2$ is the normal vector to the flutter boundary looking at the stability or divergence domains, respectively.

Finally, we assume that at the point $\mathbf{p}_0$ there exists a double positive semisimple eigenvalue λ_0 with the two linearly independent eigenfunctions u_0^1 and u_0^2. We choose them, satisfying orthonormality condition (3.30). The splitting of this eigenvalue with a change of parameters is governed by (3.32), which for $k = 2$ takes the form

$$\lambda_1^2 - \lambda_1\langle \mathbf{f}^{11} + \mathbf{f}^{22}, \mathbf{e}\rangle + \langle \mathbf{f}^{11}, \mathbf{e}\rangle\langle \mathbf{f}^{22}, \mathbf{e}\rangle - \langle \mathbf{f}^{12}, \mathbf{e}\rangle\langle \mathbf{f}^{21}, \mathbf{e}\rangle = 0, \tag{4.9}$$

where the real vectors $\mathbf{f}^{11}$, $\mathbf{f}^{12}$, $\mathbf{f}^{21}$, and $\mathbf{f}^{22}$ are defined in (3.31). The stability of the circulatory system near the point $\mathbf{p}_0$ depends on the sign of the discriminant D of quadratic equation (4.9), which can be written as

$$D = \langle \mathbf{f}^{11} - \mathbf{f}^{22}, \mathbf{e}\rangle^2 + \langle \mathbf{f}^{12} + \mathbf{f}^{21}, \mathbf{e}\rangle^2 - \langle \mathbf{f}^{12} - \mathbf{f}^{21}, \mathbf{e}\rangle^2. \tag{4.10}$$

The stability condition implies $D > 0$, while the flutter condition yields $D < 0$. The boundary between the flutter and stability domains in the parameter space is defined by the equality $D = 0$. It is easy to see from (4.10) that the equality $D = 0$ describes a circular cone $z^2 = x^2 + y^2$ in the space of three coordinates $x = \langle \mathbf{f}^{11} - \mathbf{f}^{22}, \mathbf{e}\rangle$, $y = \langle \mathbf{f}^{12} + \mathbf{f}^{21}, \mathbf{e}\rangle$, $z = \langle \mathbf{f}^{12} - \mathbf{f}^{21}, \mathbf{e}\rangle$, the flutter domain $z^2 > x^2 + y^2$ being inside the cone and the stability domain $z^2 < x^2 + y^2$ outside. The apex of the cone $(x = 0, y = 0, z = 0)$ corresponds to the double semisimple positive eigenvalue, while the skirts of the cone correspond to the double positive eigenvalues with the Keldysh chain.

Therefore, a double eigenvalue λ_0 defines a smooth surface of codimension 1 or the singularity of codimension 3 in the space of parameters $(p_1, \dots, p_n)$ depending on the number of linearly independent eigenfunctions at λ_0. This result for matrices was established in [13]. Note that eigenvalue problems with the self-adjoint operators which are widely known in physics may have only semisimple multiple eigenvalues.

Eigenvalues of multiplicity higher than 2 are also responsible for the appearance of singularities on the stability boundaries, as is seen directly from (3.13) and Figure 3.1. Indeed, the eigenvalue with multiplicity $k > 1$ splits in the nondegenerate case into k distinct complex eigenvalues implying flutter instability of the system as it was discussed in the beginning of this section. This was also shown for the discrete circulatory systems in [16].

In aeroelasticity a condition of onset of flutter, which can be easily verified, is of major importance. For circulatory systems such a condition was derived by Plaut [22] in a form $(u_0, v_0) = 0$, where u_0 and v_0 are eigenfunctions of the adjoint problems. His derivation was based on the idea that the first derivative of the nonconservative load with respect to the spectral parameter λ becomes zero at the onset of flutter. This question attracts current interest; see [23] and comments in [24]. However, Plaut's derivation is restricted by the assumptions that the flutter instability takes place due to interaction of only two eigenvalues in a one-parameter system and the nonconservative load parameter is a smooth function of λ. In our approach the "flutter condition" $(u_0, v_0) = 0$ is just a simple consequence of existence at a multiple eigenvalue λ_0 the Keldysh chain of length $k \geq 2$. Thus, this condition is satisfied at the flutter boundary of a multiparameter circulatory system.

5. Stability boundaries of the extended Beck problem. As an example of a continuous nonconservative mechanical system we consider stability of a uniform elastic cantilevered column of length L_c, loaded by a nonconservative force P; see Figure 5.1. It is assumed that the force P, which can be represented as the sum of a tangential follower force and a potential load, is acting at the free end of the column. The parameter $\eta \in [0, 1]$ measures the nonconservativity of the force P. The case $\eta = 1$ means that the column is loaded by purely tangential follower force (Beck's problem [25]). If $\eta = 0$, then the force P is potential (conservative). This problem was first considered by Dzhanelidze [26] and Kordas and Zyczkowski [27]. Note that the force P models the jet thrust acting on the free end of the column. Recent experiments on stability of such a column were carried out by Sugiyama; see [6]. We will investigate properties of the stability boundary in this problem.

Consider the transverse vibrations of the column in the plane OXY as in Figure 5.1. In the nondimensional variables

$$x = X/L_c, \quad y = Y/L_c, \quad \tau = t/\sqrt{\rho A L_c^4/EI}, \quad q = PL_c^2/EI,$$

the differential equation describing small in-plane vibrations of the column and the appropriate boundary conditions have the form

$$y''''(x,\tau) + qy''(x,\tau) + \ddot{y}(x,\tau) = 0,$$

$$y(0,\tau) = y'(0,\tau) = y''(1,\tau) = y'''(1,\tau) + (1-\eta)qy'(1,\tau) = 0.$$

Dots mean differentiation with respect to time τ, and primes denote differentiation with respect to coordinate x.

Separating time by $y(x,\tau) = u(x)\exp(i\sqrt{\lambda}\tau)$, we get the eigenvalue problem [27]

$$l(u) \equiv u'''' + qu'' = \lambda u, \tag{5.1}$$

$$\begin{aligned} &U^1(u) \equiv u(0) = 0, && U^3(u) \equiv u''(1) = 0, \\ &U^2(u) \equiv u'(0) = 0, && U^4(u) \equiv u'''(1) + (1-\eta)qu'(1) = 0. \end{aligned} \tag{5.2}$$

The corresponding adjoint eigenvalue problem looks like

$$l^*(v) \equiv v'''' + qv'' = \lambda v, \tag{5.3}$$

$$\begin{aligned} &V^1(v) \equiv -v(0) = 0, && V^3(v) \equiv v''(1) + \eta qv(1) = 0, \\ &V^2(v) \equiv v'(0) = 0, && V^4(v) \equiv -v'''(1) - qv'(1) = 0, \end{aligned} \tag{5.4}$$

and for the forms $V^5, \ldots, V^8$ we have

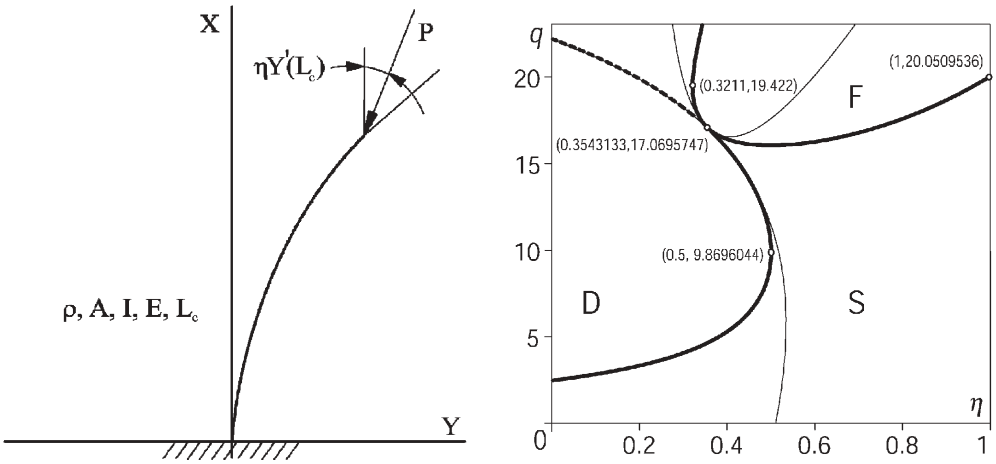

FIG. 5.1. *The extended Beck's problem and its stability diagram.*

$$(5.5)\qquad V^5 \equiv v(1), \quad V^6 \equiv -v'(1), \quad V^7 \equiv -v''(0) - qv(0), \quad V^8 \equiv v'''(0) + qv'(0).$$

Substituting the general solution of differential equation (5.1)

$$u(x) = C_1 \cosh(ax) + C_2 \sinh(ax) + C_3 \cos(bx) + C_4 \sin(bx),$$

$$a = \sqrt{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \lambda}}, \quad b = \sqrt{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \lambda}}, \quad \lambda \neq -\frac{q^2}{4},$$

into boundary conditions (5.2) we obtain the condition of the existence of a nontrivial solution $u(x)$ to eigenvalue problem (5.1), (5.2) in the form [27]

$$(5.6)\qquad \Delta(\lambda, \eta, q) = 0,$$

$$\Delta \equiv (2\lambda + (1-\eta)q^2)(1 + \cosh(a)\cos(b)) + q(2\eta - 1)(q + ab\sinh(a)\sin(b)).$$

Equation (5.6) gives eigenvalues λ, depending on the parameters η and q.

The vertical equilibrium of the column is stable if all the eigenvalues λ are positive and semisimple; i.e., each eigenvalue has the same number of eigenfunctions as its algebraic multiplicity. After substitution $\lambda = 0$ in (5.6), it gives possible values of the parameters η and q at which the system loses stability statically [26], [27],

$$(5.7)\qquad \eta(q) = \frac{\cos(\sqrt{q})}{\cos(\sqrt{q}) - 1}.$$

Equation (5.7) defines the curve of simple zero eigenvalues, the part of which forms the boundary between stability and divergence domains on the plane of parameters (η, q). The smooth parts of the flutter boundary consist of such points (η, q) that $\lambda(\eta, q)$ is a double real eigenvalue with the Keldysh chain. Calculation of the roots of characteristic equation (5.6) for different values of the load parameter q (at a fixed value of the parameter η) gives approximately the point where two simple eigenvalues form a double. Finding such points for different values of the parameter η we get the curve of double real eigenvalues.

The curves found subdivide the plane of the parameters (η, q) into stability (S), flutter (F), and divergence (D) domains; see Figure 5.1. The boundaries between these domains are shown in Figure 5.1 by the firm thick lines, while the dashed thick line shows the part of the curve of zero eigenvalues, which belongs to the divergence domain. Note that the boundary of stability domain has a singular point where the smoothness of the boundary is broken. The divergence boundary has two points with the vertical tangents. One can see that when the influence of the nonconservative part of the load q is small ($\eta < 0.5$), the column loses stability by divergence. Mainly nonconservative loads ($\eta > 0.5$) cause dynamical instability.

Our goal here is to demonstrate the advantages of the theory developed in the previous sections on the example of this stability problem. It will be applied for finding linear and quadratic approximations of the stability and instability domains both at singular and regular points of their boundaries. The explicit expression describing the overlapping of the frequency curves near the flutter boundary will be obtained and compared with the numerical results, given in [28]. Finally, we will obtain the exact coordinates of the singular point of the stability boundary, show that this point corresponds to the double zero eigenvalue with the Keldysh chain of length 2, and investigate the splitting of this eigenvalue in the vicinity of the singularity.

5.1. Bifurcation of eigenvalues in the vicinity of the flutter boundary. Consider a point $\mathbf{p}_0 = (\eta_0, q_0)$ of the flutter boundary, where the spectrum of the operator L contains double eigenvalue λ_0 with the Keldysh chain of length 2. Bifurcation of this eigenvalue is described by (4.8). Substituting the differential expression $l(u)$ from (5.1), the forms $U^1, \dots, U^4$ and $V^5, \dots, V^8$ from (5.2) and (5.5) into formula (3.14), we get the normal vector to the boundary,

$$\mathbf{f}_2 = \left(\frac{q_0 u_0'(1) v_0(1)}{\int_0^1 u_0 v_1 dx}, \frac{\int_0^1 u_0'' v_0 dx - (1-\eta_0) u_0'(1) v_0(1)}{\int_0^1 u_0 v_1 dx} \right). \tag{5.8}$$

For evaluation of the vector $\mathbf{f}_2$ it is essential to know the eigenfunctions u_0, v_0 as well as the associated functions u_1, v_1 at the double eigenvalue λ_0. The solution of eigenvalue problems (5.1), (5.2) and (5.3), (5.4) yields [5]

$$u_0(x) = \cosh(ax) - \cos(bx) + F(a\sin(bx) - b\sinh(ax)), \tag{5.9}$$

$$v_0(x) = \cosh(ax) - \cos(bx) + G(a\sin(bx) - b\sinh(ax)), \tag{5.10}$$

where the coefficients F and G depend on the parameters η and q:

$$F = \frac{a^2\cosh(a) + b^2\cos(b)}{ab(a\sinh(a) + b\sin(b))}, \quad G = \frac{(a^2+\eta q)\cosh(a) + (b^2-\eta q)\cos(b)}{b(a^2+\eta q)\sinh(a) + a(b^2-\eta q)\sin(b)}. \tag{5.11}$$

Associated function u_1 is a solution of the boundary value problem (3.4), where we should put $k = 2$ and take the differential expression and the boundary forms from (5.1) and (5.2). A particular solution of the ordinary linear differential equation with constant coefficients

$$u_1'''' + qu_1'' - \lambda_0 u_1 = u_0,$$

whose right-hand side is the linear combination of trigonometric and hyperbolic functions (5.9), has the form

$$\hat{u}_1 = x(C_1\sin(bx) + C_2\cos(bx) + C_3\sinh(ax) + C_4\cosh(ax)).$$

Substitution of $\hat{u}_1$ into the second of equations (3.4) allows one to determine the coefficients $C_1, \dots, C_4$. After these coefficients are found one tries the solution of boundary value problem (3.4) in the form

$$u_1 = \hat{u}_1 + D_1 \sin(bx) + D_2 \cos(bx) + D_3 \sinh(ax) + D_4 \cosh(ax).$$

The unknown constants $D_1, \dots, D_4$ can be found from the boundary conditions (3.4). After all necessary manipulations we arrive at the associated function u_1,

$$u_1(x) = \frac{a\sin(bx) + b\sinh(ax) + F(a^2\cos(bx) - b^2\cosh(ax))}{2ab(a^2+b^2)}x + \frac{A_1\sinh(ax) - B_1\sin(bx)}{2ab(a^2+b^2)(a\sinh(a) + b\sin(b))^2}, \tag{5.12}$$

where the coefficient F is taken from (5.11), while for the coefficients A_1, B_1 we have the expressions

$$A_1 = \frac{\sin(b)(b^2\cos(b) - a^2\cosh(a)) + 2ab\cos(b)\sinh(a)}{a^2}q + b(a^2+b^2)(1+\cosh(a)\cos(b)),$$

$$B_1 = \frac{\sinh(a)(b^2\cos(b) - a^2\cosh(a)) - 2ab\cosh(a)\sin(b)}{b^2}q + a(a^2+b^2)(1+\cosh(a)\cos(b)).$$

Similarly, solving boundary value problem (3.5) with the differential expression and boundary forms from (5.1) and (5.2) we get the associated function v_1,

$$v_1(x) = \frac{a\sin(bx) + b\sinh(ax) + G(a^2\cos(bx) - b^2\cosh(ax))}{2ab(a^2+b^2)}x + \frac{A_2\sinh(ax) - B_2\sin(bx)}{2ab(a^2+b^2)(b(a^2+\eta q)\sinh(a) + a(b^2-\eta q)\sin(b))^2}, \tag{5.13}$$

where the coefficient G is defined in (5.11) and the coefficients A_2, B_2 are

$$A_2 = q\sin(b)[[-a^2b^2 + \eta((a^2+b^2)^2 - \eta q^2)]\cosh(a) + \cos(b)(b^2-\eta q)^2]$$
$$+2qab^3(1-2\eta)\sinh(a)\cos(b) + b(a^2+b^2)[a^2b^2 + \eta^2q^2 + (a^2b^2 + \eta(1-\eta)q^2)\cos(b)\cosh(a)];$$

$$B_2 = q\sinh(a)[[a^2b^2 - \eta((a^2+b^2)^2 - \eta q^2)]\cos(b) - \cosh(a)(a^2+\eta q)^2]$$
$$-2qba^3(1-2\eta)\sin(b)\cosh(a) + a(a^2+b^2)[a^2b^2 + \eta^2q^2 + (a^2b^2 + \eta(1-\eta)q^2)\cos(b)\cosh(a)].$$

A possibility of appearance of associated functions at the critical values of the nonconservative load in Beck's problem was noticed earlier in [29]. Nevertheless, the explicit expressions for the associated functions seem to be obtained first in the present paper. Note that although the eigenfunctions u_0, v_0 are defined up to arbitrary multipliers and associated functions u_1, v_1 are defined up to the addends C_1u_0 and C_2v_0, respectively, the vector $\mathbf{f}_2$ does not depend on these uncertainties.

Consider now the point $\mathbf{p}_0 = (1, 20.0509536)$, corresponding to the double eigenvalue $\lambda_0 = 121.347049$. This point is known as critical for the column subjected to a purely tangential follower force [25]. Substituting the values of λ_0 and $\mathbf{p}_0$ into (5.9)–(5.13) we obtain the functions u_0, v_0, u_1, v_1; see Figure 5.2.

Notice that for the case of tangential force ($\eta = 1$), the eigenfunction v_0 of the adjoint eigenvalue problem has a physical meaning. It is the vibrational mode for the loss of stability of a column loaded by a force with a fixed line of action; see [1] for the theory and [6] for the experiments.

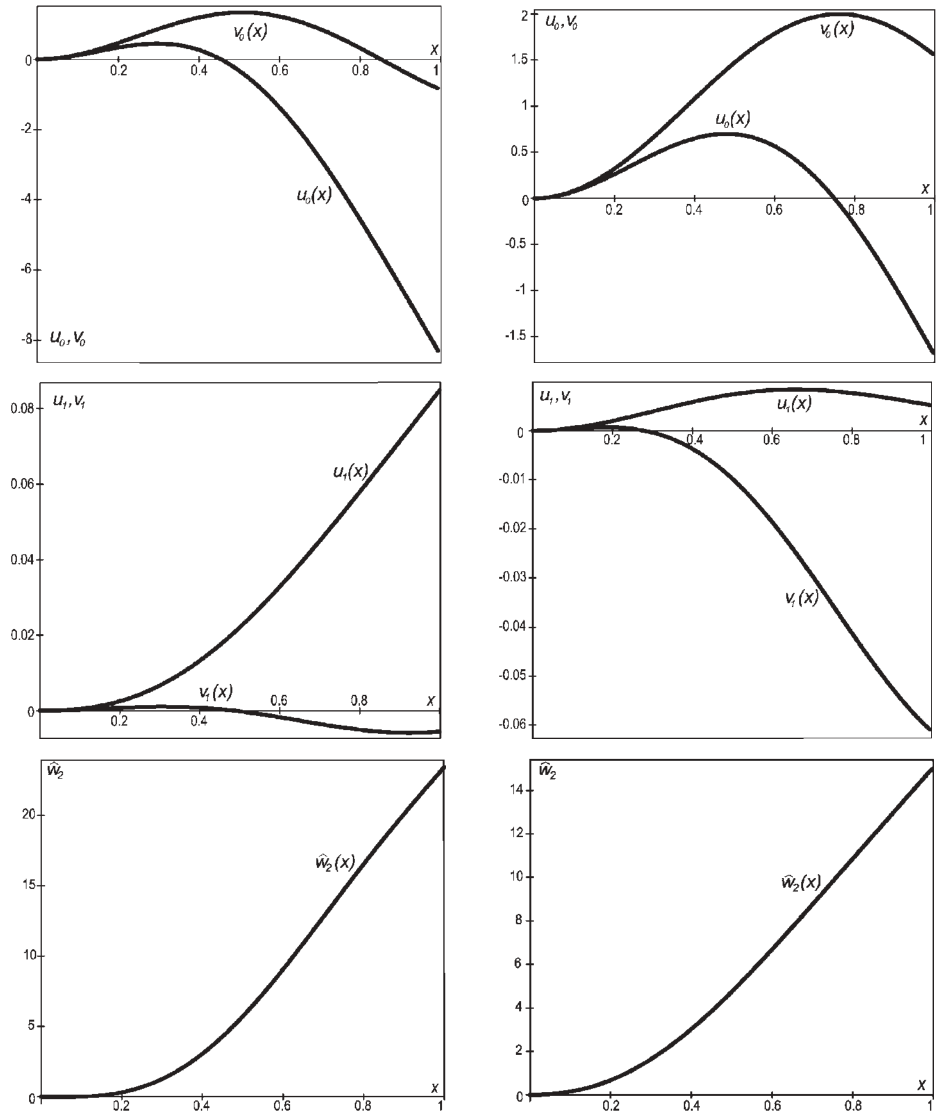

FIG. 5.2. *The eigenfunctions, associated functions, and functions* $\hat{w}_2(x)$ *at the points* $(1, 20.0509536)$ *and* $(0.35431330, 17.0695748)$*, the left and right columns, respectively.*

With the use of these functions, (5.8) gives the normal vector to the flutter boundary at the point $\mathbf{p}_0 = (1, 20.0509536)$,

$$\mathbf{f}_2 = (35458.3181, -2296.10536).$$

Let us look at the splitting of the double eigenvalue λ_0 due to change of parameters in different directions $\mathbf{e}$ on the parameter plane. Consider, for example, the vertical direction $\mathbf{e} = (0, 1)$. Taking into account that $\Delta\mathbf{p}=(0, q - q_0)$ we get from (4.8)

$$\lambda = 121.347049 \pm 47.9176936\sqrt{q_0 - q}. \tag{5.14}$$

For the horizontal variation $\Delta\mathbf{p} = (\eta - \eta_0, 0)$ corresponding to the vector $\mathbf{e} = (1, 0)$

Table 5.1
Splitting of the double eigenvalue near the point $\mathbf{p}_0 = (1, 20.0509536)$.

$(\Delta\eta, \Delta q)$	λ : Eqs. (5.14), (5.15), (5.19).	λ : Eq. (5.6)
$(0, \ \ 2\cdot10^{-5})$	$121.347049 \pm i0.21429444$	$121.342379 \pm i0.21422599$
$(0, -2\cdot10^{-5})$	121.132755	121.128319
	121.561343	121.556963
$(\ \ 2\cdot10^{-5}, 0)$	122.189169	122.188528
	120.504929	120.504432
$(-2\cdot10^{-5}, 0)$	$121.347049 \pm i0.84212016$	$121.338540 \pm i0.84201129$
$2\cdot10^{-3}\mathbf{e}_*$	120.762103	120.755389
	121.542851	121.540824

we have

$$\lambda = 121.347049 \pm 188.303792\sqrt{\eta - \eta_0}. \tag{5.15}$$

The results of probing of a small neighborhood of the point $\mathbf{p}_0$ in different directions are summarized in Table 5.1. Thus, for example, for $q = q_0 + 0.00002$, i.e., when the new point is situated above the initial point $\mathbf{p}_0$, splitting yields $\lambda = 121.347049 \pm i0.21429444$, and thus the point $\mathbf{p}_0 + \Delta\mathbf{p}$ belongs to the flutter domain; see Figure 5.1. Characteristic equation (5.6) gives for the same values of parameters two complex conjugate eigenvalues, which differ from those found with the use of (5.14) only in a sixth digit; see Table 5.1.

Degeneration condition (3.15) defines the vector $\mathbf{e}_* = (-1, -15.4428097)$ tangent to the flutter boundary at the point $\mathbf{p}_0 = (1, 20.0509536)$. The double eigenvalue λ_0 splits in the tangent direction in accordance with (3.21),

$$\lambda = \lambda_0 - \frac{a_1}{2}\epsilon \pm \frac{\epsilon}{2}\sqrt{a_1^2 - 4a_2} + o(\epsilon). \tag{5.16}$$

Substitution of the differential expression $l(u)$ from (5.1) and the forms $U^1, \dots, U^4$, $V^5, \dots, V^8$ from (5.2) and (5.5) into (3.22) and (3.23) gives the coefficients a_1 and a_2 in the form

$$a_1 = \frac{e_2^* \int_0^1 (u_0' v_1' + v_0' u_1') dx - (e_2^* \eta_0 + e_1^* q_0)(v_1(1) u_0'(1) + v_0(1) u_1'(1))}{\int_0^1 u_0 v_1 dx}, \tag{5.17}$$

$$a_2 = \frac{e_2^* \int_0^1 v_0' \hat{w}_2' dx - (e_2^* \eta_0 + e_1^* q_0) v_0(1) \hat{w}_2'(1) - e_1^* e_2^* v_0(1) u_0'(1)}{\int_0^1 u_0 v_1 dx}.$$

The functions u_0, v_0, u_1, v_1 are presented by (5.9), (5.10), (5.12), and (5.13). The function $\hat{w}_2(x)$ (Figure 5.2) is a solution of boundary value problem (3.20), where the differential expressions l_0, l_1 and forms U_0^s, U_1^s are derived from differential expression (5.1) and boundary forms (5.2) according to (3.2) and (3.3):

$$\hat{w}_2(x) = \frac{b\sin(bx) - a\sinh(ax) + Fab(\cos(bx) + \cosh(ax))}{2(a^2+b^2)} e_2^* x + \frac{A_3 \sin(bx) - B_3 \sinh(ax)}{2ab(a^2+b^2)(b\sin(b) + a\sinh(a))^2} e_2^*. \tag{5.18}$$

The coefficient F in (5.18) is defined in (5.11), and for the coefficients A_3 and B_3 we have

$$\begin{aligned} A_3 =& - a(a^2+b^2)(q + ab\sin(b)\sinh(a)) + 2a^2b(b\sinh(a)\cos(b) - a\cosh(a)\sin(b)) \\ &- 2a^3\cosh(a)(a\sinh(a) + b\sin(b)), \\ B_3 =& - b(a^2+b^2)(q + ab\sin(b)\sinh(a)) + 2b^2a(b\sinh(a)\cos(b) - a\cosh(a)\sin(b)) \\ &+ 2b^3\cos(b)(a\sinh(a) + b\sin(b)). \end{aligned}$$

With the use of the eigenfunctions, associated functions, and the function $\hat{w}_2$ we find from (5.17) the coefficients $a_1 = 194.571965$, $a_2 = -28633.4466$. Substitution of these coefficients into (5.16) gives approximate expressions for two simple eigenvalues which result from the splitting of the double λ_0 in the tangent direction to the stability boundary

$$\lambda_1 = 121.347049 - 292.473089\epsilon, \quad \lambda_2 = 121.347049 + 97.9011324\epsilon. \tag{5.19}$$

For example, take $\epsilon = 0.002$; then the double eigenvalue λ_0 splits into two positive eigenvalues (Table 5.1). This means that the tangent vector $\mathbf{e}_* = (-1, -15.4428097)$ lies in the stability domain, whence it follows that the flutter domain is convex at the point $\mathbf{p}_0$; see Figure 5.1. At the same values of the parameters, the characteristic equation has very close solutions (Table 5.1), showing thereby that formulas (5.19) give a good approximation to the directly computed eigenvalues.

Consider now the point $\mathbf{p}_0 = (0.32112653, 19.4220703)$ on the boundary between the flutter and divergence domains; see Figure 5.1. In this point there exists the negative double eigenvalue $\lambda_0 = -46.4046486$ with Keldysh chain of length 2. The normal vector to the flutter boundary evaluated at this point by formula (5.8) is

$$\mathbf{f}_2 = (-53123.691, 0).$$

The corresponding tangent vector to the boundary follows from degeneration condition (3.15),

$$\mathbf{e}_* = (0, 1).$$

One can see that the normal vector is parallel to the η-axis and is situated in the divergence domain, so the flutter boundary has a vertical tangent at the point $\mathbf{p}_0 = (0.32112653, 19.4220703)$; see Figure 5.1.

We are interested now in behavior of the frequency curves $\omega(q)$, where $\omega=\sqrt{\lambda}$ is a frequency of oscillations, in the vicinity of the point $\mathbf{p}_0$ of the boundary between the flutter and divergence domains. From the formula (3.21) it follows that along the curves $\mathbf{p} = \mathbf{p}_0 + \epsilon\mathbf{e}_* + \epsilon^2\mathbf{d} + o(\epsilon^2)$ the double eigenvalue splits according to

$$(\lambda - \lambda_0)^2 + \langle \mathbf{h}, \mathbf{e}_* \rangle (\lambda - \lambda_0)\epsilon + \langle \mathbf{H}\mathbf{e}_*, \mathbf{e}_* \rangle \epsilon^2 = \epsilon^2 \langle \mathbf{f}_2, \mathbf{d} \rangle + o(\epsilon^2); \tag{5.20}$$

see [30]. Taking into account that along the curve $\mathbf{p}(\epsilon)$ tangent to the flutter boundary at the point $\mathbf{p}_0$,

$$q - q_0 = \epsilon e_*^q + o(\epsilon), \quad \eta - \eta_0 = \epsilon e_*^\eta + \epsilon^2 d^\eta + o(\epsilon^2),$$

we convert expression (5.20) into

$$\left(\lambda - \lambda_0 + \frac{h^q}{2}(q - q_0)\right)^2 - \left(\frac{(h^q)^2}{4} - H^{qq}\right)(q - q_0)^2 = f_2^1(\eta - \eta_0). \tag{5.21}$$

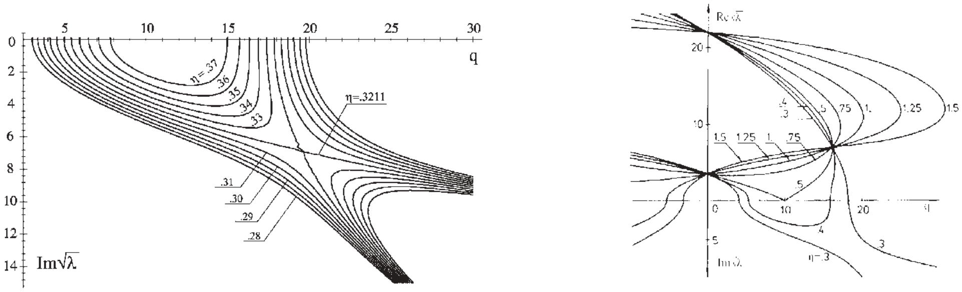

FIG. 5.3. *The frequency curves (from left to right) by* (5.21) *and by the work of* [28].

Formulas (3.22)–(3.25) give the components of the vector **h** and matrix **H**,

$$h^{\eta} = 686267882692882, \quad h^{q} = 32.1039479,$$

$$H^{\eta\eta} = 0, \quad H^{\eta q} = 1917.18297, \quad H^{qq} = 93.4817323.$$

The double eigenvalue λ_0 does not split in the first approximation if the discriminant of (5.21) is zero. This condition gives us the quadratic approximation of the flutter boundary near the point $\mathbf{p}_0 = (0.32112653, 19.4220703)$,

$$\eta = 0.32112653 + 0.0030906(q - 19.4220703)^2. \tag{5.22}$$

Equation (5.22) shows that the flutter domain is convex at the point $\mathbf{p}_0$; see Figure 5.1. Formula (5.21) approximates in the vicinity of the point q_0, λ_0 the family of frequency curves $\omega(q) = \sqrt{\lambda(q)}$, parameterized by η. At $\eta = \eta_0$, (5.21) disintegrates into two parts:

$$q = 0.30878129(Im\sqrt{\lambda})^2 + 5.09318321, \quad q = 0.03464354(Im\sqrt{\lambda})^2 + 17.814449. \tag{5.23}$$

Parabolas (5.23) are symmetrical with respect to the axis q and are situated on the plane $(q, Im\sqrt{\lambda})$. At the points $(19.4220703, \pm 6.81209576)$ corresponding to two purely imaginary eigenfrequencies $\omega = \pm i6.81209576$ these parabolas intersect.

In the left picture of Figure 5.3, the behavior of frequency curves described by (5.21) near one of the intersecting points is shown. One can see that at $\eta < \eta_0$ there exist two purely imaginary frequencies, meaning static instability. With the increase of η, frequency curves come closer together, overlap, and at $\eta > \eta_0$ move apart, forming a zone of complex eigenvalues (flutter). In the right picture of Figure 5.3, the dependence of the two lowest eigenfrequencies $\omega = \sqrt{\lambda}$ on the load q at the different values of the parameter $\eta \in [0.3, 1.5]$, obtained earlier in [28] by numerical solution of characteristic equation (5.6), is shown. Comparing the two pictures of Figure 5.3 we note a good qualitative and quantitative agreement in behavior of frequency curves calculated by two different methods in the range $\eta \in [0.3, 0.4]$.

5.2. Behavior of eigenvalues near the stability-divergence boundary. Consider a point $\mathbf{p}_0 = (\eta_0, q_0)$ on the boundary between the stability and divergence domains, where the spectrum of the eigenvalue problem (5.1), (5.2) contains a simple eigenvalue $\lambda_0 = 0$. Due to variation of parameters, a simple eigenvalue changes according to formula (4.7). Substituting the differential expression $l(u)$ from (5.1)

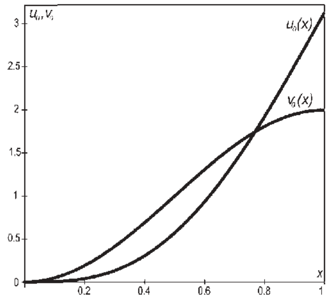

FIG. 5.4. *The eigenfunctions of the zero eigenvalue at the point* $\mathbf{p}_0 = (0.5, 9.86960440)$.

TABLE 5.2
Changing of zero eigenvalue near the point $\mathbf{p}_0 = (0.5, 9.86960440)$.

$(\Delta\eta, \Delta q)$	λ : Eq. (5.27)	λ : Eq. (5.6)
$(\ 10^{-4}, 0)$	0.00789568	0.00789498
$(-10^{-4}, 0)$	−0.00789568	−0.00789639

and the forms $U^1, \ldots, U^4$ and $V^5, \ldots, V^8$ from (5.2) and (5.5) into (3.14), we get the normal vector $\mathbf{f}_1$,

$$\mathbf{f}_1 = \left(\frac{q_0 u_0'(1) v_0(1)}{\int_0^1 u_0 v_0 dx}, \frac{\int_0^1 u_0'' v_0 dx - (1-\eta_0) u_0'(1) v_0(1)}{\int_0^1 u_0 v_0 dx} \right). \tag{5.24}$$

The eigenfunctions u_0 and v_0 at the simple zero eigenvalue have the form

$$u_0 = \sin(b) - xb\cos(b) - \sin(b)\cos(bx) + \cos(b)\sin(bx), \tag{5.25}$$

$$v_0 = 1 - \cos(bx), \quad b = \sqrt{q_0}. \tag{5.26}$$

These eigenfunctions are solutions of eigenvalue problems (5.1)–(5.4) at $\lambda_0 = 0$ and are presented in Figure 5.4.

Consider the point $\mathbf{p}_0 = (0.5, 9.86960440)$ on the divergence boundary described by (5.7). Substituting eigenfunctions (5.25) and (5.26) evaluated at this point into (5.24), we get the normal vector to the divergence boundary,

$$\mathbf{f}_1 = (78.9568352, 0).$$

Hence, the divergence boundary has the vertical tangent at the point $\mathbf{p}_0$; see Figure 5.1. Variation of the parameters $\Delta\mathbf{p} = (\eta - \eta_0, 0)$ changes the zero eigenvalue. According to (4.7) we have

$$\lambda = 78.956835(\eta - \eta_0). \tag{5.27}$$

One can see that for $\eta - \eta_0 < 0$ the eigenvalue $\lambda_0 = 0$ becomes negative. Therefore, the point $\mathbf{p}_0 + \Delta\mathbf{p}$ is inside the divergence domain; see Figure 5.1. If $\eta - \eta_0 > 0$, we come to the stability domain; see Table 5.2.

5.3. The singularity 0^2 of the stability boundary. Figure 5.1 clearly shows that the flutter domain has a common boundary with the domains of stability and divergence. Recall that at the points of the boundary between flutter and stability domains, the spectrum of the differential operator contains positive double eigenvalues, while at the points of the boundary between the flutter and divergence domains, double eigenvalues are negative.

Thus, the double eigenvalue becomes double zero at such point of the flutter boundary that separates stability and divergence domains. At the same time the point with double zero eigenvalue should belong to the curve of zero eigenvalues (5.7). Besides, due to (3.6) the orthogonality condition $\int_0^1 u_0 v_0 dx = 0$ must be true at the points of the flutter boundary. It is clear that this integral evaluated at the points of curve (5.7) becomes zero only at the point corresponding to the double zero eigenvalue.

Integrating the product of the eigenfunctions $u_0(x)$ and $v_0(x)$ from (5.25) and (5.26) over the range $[0, 1]$, we come to the transcendental equation for the ordinate of the desired point,

$$q_0 = (\sqrt{q_0} - 2\sin(\sqrt{q_0}))(\sqrt{q_0}(1 + 2\cos(\sqrt{q_0})) - 4\sin(\sqrt{q_0})). \tag{5.28}$$

The minimal element of the set of solutions of (5.28) at $q_0 > 0$ is $q_0 = 17.0695748$. Substituting this solution into (5.7), we find the corresponding value of the second parameter $\eta_0 = 0.35431330$.

Note that an equation similar to (5.28) was derived first in [27] from the analysis of characteristic equation (5.6) and without use of the eigenfunctions. However, formula (3.23) of [27] contains a misprint: the first term $k_2^2 l^2 \cos^2 k_2 l$ should be read as $k_2^2 l^2 \cos k_2 l$. Nevertheless, the coordinates of the singular point found in [27] are correct and coincide with those obtained from (5.28).

Thus, at the point $\mathbf{p}_0 = (0.35431330, 17.0695748)$ there exists the double eigenvalue $\lambda_0 = 0$ with the Keldysh chain of length 2. Following Arnold [13] we denote this point by the symbol 0^2, where the upper index means the length of the Keldysh chain corresponding to the double zero eigenvalue.

The bifurcation of a double eigenvalue is described by formula (4.8). To evaluate the normal vector $\mathbf{f}_2$ at this point, one needs to know the associated functions u_1, v_1 at the double zero eigenvalue along with the eigenfunctions u_0, v_0. Solving at $k = 2$ and $\lambda = 0$ boundary value problems (3.4) and (3.5) with the differential expressions and boundary forms from (5.1)–(5.4) we get

$$u_1 = -\frac{\cot(b)}{6b}x^3 + \frac{1}{2b^2}x^2 + \frac{\cot(b)(\cos(bx) - 1) + \sin(bx)}{2b^3}x + \frac{(bx - \sin(bx))(b + 2b\cos(b) - 2\sin(b))}{2b^4 \sin^2(b)}, \tag{5.29}$$

$$v_1 = \frac{x + x^2}{2b^2} + \frac{x - 1}{2b^3}\sin(bx) + \frac{b^2\cos(b) - \sin^2(b)}{b^4(b\cos(b) - \sin(b))}(\sin(bx) - bx), \tag{5.30}$$

where $b = \sqrt{q_0}$.

Substituting eigenfunctions (5.25) and (5.26) and associated function (5.30) into expression (5.8), we find the normal vector to the flutter boundary at the point $\mathbf{p}_0$ $= (0.35431330, 17.0695748)$,

$$\mathbf{f}_2 = (-24288.8139, -1024.49949). \tag{5.31}$$

TABLE 5.3
Splitting of the double zero near the singular point $\mathbf{p}_0 = (0.35431330, 17.0695748)$.

$(\Delta\eta, \Delta q)$	λ : Eqs. (5.32), (5.33)	λ : Eq. (5.6)
$(0,\ 10^{-4})$	$\pm i0.32007804$	$-0.00151188 \pm i0.32007586$
$(0, -10^{-4})$	0.32007804 -0.32007804	0.32159210 -0.31856833
$(\ 10^{-4}, 0)$	$\pm i1.55848689$	$0.02668744 \pm i1.55823291$
$(-10^{-4}, 0)$	1.55848689 -1.55848689	1.53205170 -1.58543004
$-10^{-5}\mathbf{e}_*$	-0.01207531 -0.00043108	-0.01207543 -0.00043108
$10^{-5}\mathbf{e}_*$	0.01207531 0.00043108	0.01207520 0.00043108

Knowing the normal vector allows us to study the neighborhood of the point of the flutter boundary in any direction $\mathbf{e}$ such that $\langle \mathbf{f}_2, \mathbf{e} \rangle \neq 0$. In particular, for two orthogonal directions $\mathbf{e} = (1, 0)$ and $\mathbf{e} = (0, 1)$, we get

$$\lambda = \pm 155.848689\sqrt{\eta_0 - \eta}, \quad \lambda = \pm 32.0078037\sqrt{q_0 - q}, \tag{5.32}$$

appropriately. It is easy to see that in the typical situation the double zero eigenvalue splits either into a complex conjugate pair or into two real eigenvalues, one of which is negative; see Table 5.3. Thus, the normal vector $\mathbf{f}_2$ at the point $\mathbf{p}_0$ is directed into the divergence domain. The inequality $\langle \mathbf{f}_2, \mathbf{e} \rangle > 0$ defines the tangent cone to this domain, and $\langle \mathbf{f}_2, \mathbf{e} \rangle < 0$ defines the tangent cone to the flutter domain; see Figure 5.1. Only curves, emitted in the tangent direction to the boundary, can reach the stability domain from the singular point.

Using the degeneration condition $\langle \mathbf{f}_2, \mathbf{e}_* \rangle = 0$, we find the tangent vector $\mathbf{e}_* = (1, -23.7079804)$. To examine whether this vector points to the stability domain, we should consider bifurcation of a double zero eigenvalue in the degenerate case. Substituting eigenfunctions (5.25) and (5.26), associated functions (5.29) and (5.30), and the function $\hat{w}_2$, which according to (5.18) takes the form

$$\hat{w}_2 = e_2^* x \frac{\cot(b)(\cos(bx) - 1) + \sin(bx)}{2b} + e_2^* \frac{bx - \sin(bx)}{2b\sin^2(b)}, \quad b = \sqrt{q_0},$$

into expressions (5.17) we find the coefficients of (5.16),

$$a_1 = 1250.63981, \quad a_2 = 52054.6889.$$

In accordance with (5.16) in the first approximation we have

$$\lambda_1 = 1207.53146\epsilon, \quad \lambda_2 = 43.1083501\epsilon. \tag{5.33}$$

It follows from (5.33) that the double zero eigenvalue splits into two positive simple eigenvalues (stability) only if the parameters change in the direction specified by the vector $\mathbf{e}_* = (1, -23.7079804)$; see Table 5.3. Changing the parameters in the opposite direction results in the splitting of the double $\lambda_0 = 0$ into two negative simple eigenvalues, which means static instability (divergence). Note that the approximate expressions for the eigenvalues are in a good agreement with the solutions of characteristic equation (5.6); see Table 5.3.

One can see that the tangent cone to the stability domain at the singular point is a ray on the plane of parameters. Stability domain in the vicinity of this point is a long narrow tongue (Figure 5.1). Our technique allows us to find the quadratic approximation of the flutter and divergence domains and therefore the stability domain near the singular point.

It is easy to see that (5.20), describing splitting of the double eigenvalue $\lambda_0 = 0$ along smooth curves tangent to the flutter boundary at the point $\mathbf{p} = \mathbf{p}_0$, can be rewritten as follows [30]:

$$\lambda^2 + \langle \mathbf{h}, \Delta\mathbf{p}\rangle\lambda + \langle \mathbf{H}\Delta\mathbf{p}, \Delta\mathbf{p}\rangle = \langle \mathbf{f}_2, \Delta\mathbf{p}\rangle + o(||\Delta\mathbf{p}||^2). \tag{5.34}$$

Components of the real vector $\mathbf{h}$ and real symmetrical matrix $\mathbf{H}$ are determined by formulas (3.22)–(3.25). Their evaluation at the singular point gives

$$\begin{gathered} h^\eta = -917.197355, \quad h^q = 14.0645660, \\ H^{\eta\eta} = 0, \quad H^{\eta q} = -690.854898, \quad H^{qq} = 34.3323737. \end{gathered} \tag{5.35}$$

Equation (5.34) provided that $\lambda = 0$ gives the quadratic approximation of the divergence boundary near the singular point,

$$f_2^\eta(\eta - \eta_0) + f_2^q(q - q_0) = 2H^{\eta q}(\eta - \eta_0)(q - q_0) + H^{qq}(q - q_0)^2. \tag{5.36}$$

The equality of the discriminant of (5.34) to zero guarantees the nonsplitting of the double zero eigenvalue and therefore defines the approximation of the flutter boundary

$$\begin{gathered} f_2^\eta(\eta - \eta_0) + f_2^q(q - q_0) \\ = (h^\eta(\eta - \eta_0) + h^q(q - q_0))^2/4 - (2H^{\eta q}(\eta - \eta_0)(q - q_0) + H^{qq}(q - q_0)^2). \end{gathered} \tag{5.37}$$

Substitution of the components of the normal vector $\mathbf{f}_2$ from (5.31) and the vector $\mathbf{h}$ and matrix $\mathbf{H}$ from (5.35) into (5.36) and (5.37) gives the quadratic approximations of the flutter and divergence domains in the vicinity of the point $\mathbf{p}_0 = (0.35431330, 17.0695748)$. These approximations are shown in Figure 5.1 by the thin solid lines. One can see that the approximation of the divergence domain is very good at far distances from the singular point while the approximation of the flutter domain is good only in the neighborhood of the point $\mathbf{p}_0$.

6. Conclusion. A new approach to obtain explicit formulas for the bifurcation of multiple eigenvalues of non-self-adjoint differential operators smoothly dependent on a vector of real parameters is presented. The formulas found use the derivatives of the differential expression and the boundary forms with respect to parameters as well as the functions of the Keldysh chain evaluated at the point of the parameter space corresponding to a multiple eigenvalue.

The results obtained let us study the splitting of the multiple eigenvalues in both regular and degenerate cases and serve as a basis for the sensitivity analysis of continuous nonconservative systems. This allows one to avoid the variational calculus in every specific problem to find sensitivities of eigenvalues or critical values of parameters.

Then the multiparameter stability problems of continuous circulatory systems are studied. It is found that the stability boundaries of these systems are smooth surfaces in the parameter space corresponding to simple zero (divergence) or double real eigenvalues with Keldysh chain of length 2 (flutter). It is shown that the flutter condition for the circulatory systems is a simple consequence of the existence of the Keldysh chain of length $k \geq 2$.

The advantages of the proposed approach are illustrated by the mechanical example known as the extended Beck problem. With the use of the bifurcation analysis of eigenvalues, stability boundaries in this problem are investigated. Linear and quadratic approximations to the stability and instability domains at both regular and singular points of their boundaries are found and compared with the exact numerical values.

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