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A Coriolis force in an inertial frame

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Abstract

Particles in rotating saddle potentials exhibit precessional motion which, up to now, has been explained by explicit computation. We show that this precession is due to a hidden Coriolis-like force which, unlike the standard Coriolis force, is present in the inertial frame. We do so by finding a hodograph-like 'guiding center' transformation using the method of normal form. We also point out that the transformation cannot be of contact type in principle, thus showing that the standard (in applied literature) heuristic averaging obscures the fact that the transformation of the position *must* involve the velocity.

Keywords: Coriolis force, rotating potentials, contact transformations
Mathematics Subject Classification numbers: 37J25, 37J40, 70F99

(Some figures may appear in colour only in the online journal)

1. Introduction and background

We consider the motion of a particle in the rotating saddle potential in the plane. The 'spinning' potential whose graph is obtained by rotating the graph of a fixed potential $U_0(x) = U_0(x_1, x_2)$ with angular velocity ω is

$$U(x, t) = U_0(R^{-1}x), \quad x = (x_1, x_2) \in \mathbb{R}^2$$

where

$$R = R(\omega t) = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}$$

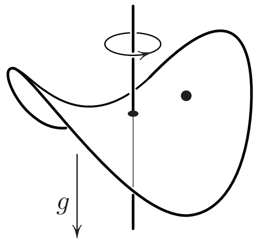

Figure 1. A particle on a rotating saddle surface.

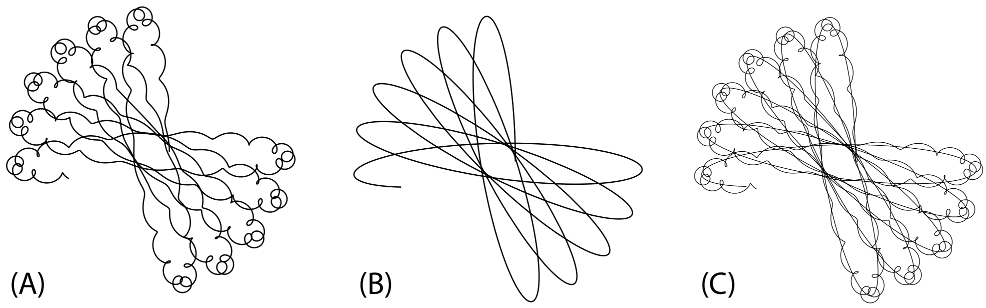

Figure 2. A typical trajectory x; its 'guiding center' u; their superposition.

is the counterclockwise rotation. If U_0 is a saddle with equal principal curvatures, then

$$U_0(x) = \frac{1}{2}(x_1^2 - x_2^2)$$

without the loss of generality, and the equations of motion $\dot{x} = -\nabla U$ take the form

$$\ddot{x} + S(\omega t)x = 0, \quad x \in \mathbb{R}^2, \tag{1}$$

where

$$S(\tau) = \begin{pmatrix} \cos 2\tau & \sin 2\tau \\ \sin 2\tau & -\cos 2\tau \end{pmatrix}, \quad \tau = \omega t.$$

These equations describe the linearized motion of a particle sliding without friction on a rotating saddle surface (with equal and opposite principal curvatures) in the presence of gravity ($m = 1$, $g = 1$), figure 1. It is a surprising fact, known for almost a century [1–3, 5–7] that the equilibrium position of the particle becomes stable if the surface rotates around the vertical axis sufficiently fast (a heuristic explanation of this effect can be found in [8], and is also given below). Numerical experiments show another puzzling effect: the Foucault-like precession [8–11], figure 2.

A similarly surprising phenomenon is the stabilization of a ball rolling without slipping on a rotating saddle surface (several demonstrations can be found on YouTube) [10, 12]. Superficially, the two effects appear to be the same; however, the reasons for stability are entirely different. For the rolling ball, the gyroscopic effect, which has no counterpart for a point mass in our case, plays the key role. In fact, the rolling ball is stable even if the surface is horizontal and flat, [13]. The rolling ball is an entirely different system: first, it is a nonholonomic system (see [14, 15] for more details), unlike the one considered here, and second, it has more degrees of freedom.

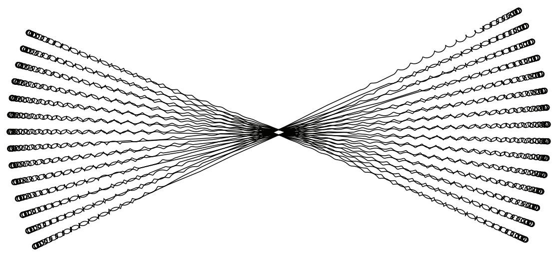

Figure 3. The 'Coriolis' effect is one order in ε higher than the stabilizing effect: here $\varepsilon = 0.2$ versus $\varepsilon = 0.5$ in figure 2.

Returning to the particle in the rotating saddle potential, the force field $-S(\omega t)x$ in (1) admits the following nice interpretation. Consider the saddle force field $\langle -x_1, x_2\rangle$, and make it time-dependent by rotating each vector counterclockwise with angular velocity 2ω. Equation (1) describes the motion of the unit point mass in this force field.

2. The results

2.1. An averaging result

Theorem 1. *Let $x(t)$ be a solution of (1), and consider the associated function*

$$u = x - \frac{\varepsilon^2}{4} S(t/\varepsilon)(x - \varepsilon J\dot{x}), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

built out of x and $\dot{x}$. This 'image' of x satisfies

$$\ddot{u} - \frac{\varepsilon^3}{4} J\dot{u} + \frac{\varepsilon^2}{4} u = \varepsilon^4 f(u, \dot{u}, \varepsilon), \tag{3}$$

where f is linear in u, $\dot{u}$ and analytic in ε, in a fixed neighborhood of $\varepsilon = 0$.

Figure 2 shows a typical trajectory of x, the corresponding trajectory of u, and their superposition.

According to (3), the 'guiding center' u behaves (modulo the $O(\varepsilon^4)$-terms) as a unit point mass with a unit charge, subject to the restoring force $-\frac{\varepsilon^2}{4}u$ and to the magnetic force due to the magnetic field of constant magnitude $B = \varepsilon^3/4$ perpendicular to the u-plane. Alternatively, one can think of $-(\varepsilon^3/4)\, J\dot{u}$ as the Coriolis force which would have been caused by the rotation of the reference frame with angular velocity $\varepsilon^3/8$, although of course our frame is inertial. This seems to be the first example of a Corilois-like force arising in an inertial reference frame [19].

It should be noted that the $O(\varepsilon^3)$ 'magnetic' effect is small compared to the $O(\varepsilon^2)$ restoring force in (3), as reflected by the 'petals' in figure 3 becoming closely spaced for smaller ε.

While a heuristic explanation of the stabilizing restoring force $-\frac{\varepsilon^2}{4}u$ is known (and described in section 3), such an explanation of the 'Coriolis force' $\frac{\varepsilon^3}{4}\dot{u}$ is still unknown. Part of the difficulty lies in the fact that u, unlike x, is not a 'material particle' but rather a mixture of position and velocity.

Remark 1. It is tempting to study the system in the rotating frame (as had been done, [8, 10]), since in such a frame the equations become autonomous. The problem with this approach is that solutions such as those shown in figure 2 move rapidly relative to the rapidly rotating frame and thus the region in the phase space they occupy grows with $\omega = \varepsilon^{-1}$. In fact, in the rotating frame one can write down explicit solutions ([1–3, 7, 19, 20]), but this hides the Coriolis effect.

Theorem 1 was announced in an expository article [16], but without the derivation, and more importantly without an explanation of how it was arrived at, and without the mention of an obstruction that (for us) stood in the way of obtaining this result (theorem 2 in the next section).

2.2. Contact transformations

The standard (in the physics and applied mathematics literature) textbook heuristic averaging procedure as described in Landau–Lifshitz [17–19] involves transformations acting on the position variable, with the velocity transformation dictated by the transformation of the position. In other words, these are transformations of contact type. In attempting to come up with an averaging result we ran into difficulties which led to the realization that this class of contact transformations is actually too narrow to eliminate any higher order terms beyond constant, not only for (1), but even in the simpler scalar case $\ddot{x} = a(\omega t)f(x)$, where $a : \mathbb{R} \to \mathbb{R}$ is periodic and $f : \mathbb{R}^n \to \mathbb{R}^n$, considered in [18]. The heuristic procedure used in [18] does give a correct differential equation in the end, but it does not make it clear that the transformation *must* involve the velocity, and not only the position, and thus is not justified, except for the fact that it gives the correct result. In fact, the transformation used in the proof in [29] does involve the velocity, i.e. is not of contact type, although this fact also was not stated.

We now make this precise as follows.

Our extended phase space $\{x, y, t\} = \mathbb{R}^5$ carries a natural contact structure [4])

$$\alpha = \mathrm{d}x - y\,\mathrm{d}t, \tag{4}$$

and the velocity vectors

$$V = \langle \dot{x}, \dot{y}, \dot{t} \rangle = \langle\, y, -S(\omega t)x, 1 \rangle$$

corresponding to (1) belong to the distribution $\alpha = 0$ defined by this structure since $\alpha(V) = \dot{x} - y = 0$.

Now a general contact transformation of $\mathbb{R}^{2n+1}$ ($n = 2$ in our case) preserving the contact structure $\alpha = 0$ and not involving change of t (i.e. preserving the planes $t = \text{const.}$) is determined by a single map $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ acting on $\{(x, t)\}$ via

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \begin{pmatrix} \varphi \\ \varphi_x y + \varphi_t \\ t \end{pmatrix}, \tag{5}$$

where $\varphi = \varphi(x, t)\,(\varphi(0, t) = 0)$.

Theorem 2. *No contact transformation, i.e. no map of the form (5), i.e. no substitution of the form $x = \varphi(u, t)$ can eliminate time-dependence in the terms of the order $O(\varepsilon)$ in the system (1).*

A complete proof of this theorem is given at the end, but the gist can alternatively be seen as follows. Substitute (2) into (3); rather than going through tedious manipulations, let us simply consider all possible sources in (2) which contribute $O(\varepsilon)$-terms to (3) and see how they can cancel each other, as (3) implies they do. One such source is the cubic term $\frac{\varepsilon^3}{4}SJ\dot{x}$ in (2);

since $\dot{S} = O(\varepsilon^{-1})$, two differentiations produce precisely one $O(\varepsilon)$ term: $\frac{\varepsilon^3}{4}\ddot{S}J\dot{x}$. The only other source of $O(\varepsilon)$-terms is $\frac{\varepsilon^2}{4}Sx$, which after two differentiations contributes $2\frac{\varepsilon^2}{4}\dot{S}\dot{x}$. This latter contribution could not have been canceled without the cubic term containing $\dot{x}$ in (2), suggesting that the latter term is indeed necessary.

3. A heuristic explanation of stability

For a fixed location x_0, the force vector

$$F_0(t) = -S(t/\varepsilon)x_0 \tag{6}$$

rotates counterclockwise with angular velocity $2\varepsilon^{-1} = 2\omega$. Were the force independent of x near x_0, a non-drifting particle would have moved in a circle, with $F_0(t)$, the centripetal force, related to the radius vector $r = x - x_0$ via $F = -(2\omega)^2 r = -(4/\varepsilon^2)(x - x_0)$, so that

$$x = x_0 - \frac{\varepsilon^2}{4}F = x_0 + \frac{\varepsilon^2}{4}Sx_0. \tag{7}$$

Substituting this for x_0 in (6) gives a better approximation for the force:

$$F_1(t) = -S\left(x_0 + \frac{\varepsilon^2}{4}Sx_0\right) = -Sx_0 - \frac{\varepsilon^2}{4}x_0, \tag{8}$$

showing that in this approximation (improving on (6)) $F_1(t)$ describes a *shifted* circle, so that the average force (in this approximation) is

$$\overline{F_1(t)} = -\frac{\varepsilon^2}{4}x_0;$$

this is precisely the restoring force in (3).

To repeat the above in a different way, consider the force $F = -S(t/\varepsilon)(x_0 + r(t))$ changing with t, as $r(t)$ travels in a small circle counterclockwise with the angular velocity $2/\varepsilon$, so that the force

$$F = -S(t/\varepsilon)(x_0 + r(t)) = -Sx_0 - Sr.$$

The key point now is that the last term Sr is constant and points into the origin (thus explaining the restoring force). Indeed, while r rotates counterclockwise, the time-dependence of S precisely cancels this rotation (according to the last remark in section 3). Thus the average $\overline{F} = -Sr = -r(0) = -\frac{\varepsilon^2}{4}x_0$, reconfirming the earlier conclusion.

4. Rotating saddle in physical applications

Equation (1), as well as their autonomous version in the rotating frame, arise in numerous applications across many seemingly unrelated branches of classical and modern physics [6, 7, 20]. In celestial mechanics the rotating saddle equations describe linear stability of the triangular Lagrange libration points L_4 and L_5 in the restricted circular three-body problem [22, 23]. For this reason the classical work by Gascheau of 1843 may be considered as the first one that established stability conditions for a particle on a rotating saddle [21, 24]. However, it was not until Brouwer, one of the authors of the fixed point theorem in topology, considered in 1918 stability of a heavy particle on a rotating slippery surface [1–3] that the rotating saddle trap per se became an object for investigation.

Indeed, according to Earnshaw's theorem an electrostatic potential cannot have stable equilibria, i.e. minima, since such potentials are harmonic functions. The theorem does not apply, however, if the potential depends on time; in fact, the 1989 Nobel Prize in physics was awarded to Paul [25] for his invention of the trap for suspending charged particles in an oscillating electric field. Paul's idea was to stabilize the saddle by 'vibrating' the electrostatic field, by analogy with the so-called Stephenson–Kapitsa pendulum [17, 18, 26–29] in which the upside-down equilibrium is stabilized by vibration of the pivot. Brouwer [1, 2] explicitly demonstrated that, instead of vibration, the saddle can also be stabilized by rotation of the potential (in two dimensions). This effect is used, e.g. in quantum optics, in the theory of rotating radio-frequency ion traps [8] and for guiding electrons inside Bessel beams of an electromagnetic field [11].

In plasma physics equation (1) appear in the modeling of a stellatron—a high-current betatron with stellarator fields used for accelerating electron beams in helical quadrupole magnetic fields [9, 19, 30, 31]. In atomic physics the stable triangular Lagrange points were produced in the Schrödinger–Lorentz atomic electron problem by applying a circularly polarized microwave field rotating in synchrony with an electron wave packet in a Rydberg atom [22]. This has led to a first observation of a non-dispersing Bohr wave packet localized near the Lagrange point while circling the atomic nucleus indefinitely [32]. Recently, the rotating saddle equation (1) reappeared in the study of confinement of massless Dirac particles, e.g. electrons in graphene [33]. Even stability of a rotating flow of a perfectly conducting ideal fluid in an azimuthal magnetic field possesses a mechanical analogy with the stability of Lagrange triangular equilibria and, consequently, with the gyroscopic stabilization on a rotating saddle [34]. Finally, we note that in mechanical engineering equation (1) describes stability of a mass mounted on a non-circular weightless rotating shaft subject to a constant axial compression force [5, 35].

5. Proofs

5.1. Proof of theorem 1

In an attempt to bring (1) to a normal form, let us choose a new variable $x_1 \in \mathbb{R}^2$ via

$$x = x_1 + \frac{\varepsilon^2}{4} S(t/\varepsilon) x_1. \tag{9}$$

This transformation is suggested by the heuristic discussion in section 3. The transformation (9) converts (1) into

$$\ddot{x}_1 - \varepsilon S J \dot{x}_1 + \frac{\varepsilon^2}{4} x_1 + \frac{\varepsilon^3}{4} J \dot{x}_1 - \frac{\varepsilon^4}{16} S x_1 = O(\varepsilon^5). \tag{10}$$

Remark 2. The fact that the average $\overline{S} = 0$ (a zero matrix) may suggest that the averaging of (10) may give

$$\ddot{x}_2 + \frac{\varepsilon^2}{4} x_2 + \frac{\varepsilon^3}{4} J \dot{x}_2 = O(\varepsilon^5);$$

interestingly, this coincides with the correct result (3) *except* for the sign in front of the Coriolis term.

Proof of (10) goes by direct substitution, and we omit the details, pointing out only that the identities

$$\dot{S} = \frac{\mathrm{d}}{\mathrm{d}t} S(t/\varepsilon) = -2\varepsilon^{-1} SJ \tag{11}$$

and

$$\ddot{S} = -4\varepsilon^{-2} S \tag{12}$$

should be used in the process.

We expect, according to theorem 2, that the transformations must involve both x and $\dot{x}$, and we therefore write (10) as a system

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = \varepsilon SJy_1 - \dfrac{\varepsilon^2}{4} x_1 - \dfrac{\varepsilon^3}{4} Jy_1 + \dfrac{\varepsilon^4}{16} Sx_1 + O(\varepsilon^5), \end{cases}$$

or, more compactly as a system in $\mathbb{R}^4$:

$$\dot{z}_1 = (A_0 + \varepsilon A_1(t) + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \varepsilon^4 A_4 + O(\varepsilon^5))\, z_1, \tag{13}$$

where

$$z_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ A_0 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \ A_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & SJ \end{pmatrix}, \tag{14}$$

$$A_2 = -\frac{1}{4} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \ A_3 = -\frac{1}{4} \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}, \ A_4 = \frac{1}{16} \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}. \tag{15}$$

5.1.1. Averaging the $O(\varepsilon)$-term. We seek to kill time-dependence in the $O(\varepsilon)$-term in (13) via the change of variables

$$z_1 = (I + \varepsilon^2 T_1) z_2, \ \ T_1 = T_1(t/\varepsilon), \tag{16}$$

where $T_1(\tau)$ is a periodic 4×4 matrix function of period π. Substituting this into (13) and using

$$(I + \varepsilon^2 T_1)^{-1} = I - \varepsilon^2 T_1 + \varepsilon^4 T_1^2 + O(\varepsilon^6),$$

we obtain

$$\dot{z}_2 = (B_0 + \varepsilon B_1(t) + \varepsilon^2 B_2 + \varepsilon^3 B_3 + \varepsilon^4 B_4 + O(\varepsilon^5)) z_2, \tag{17}$$

where

$$B_0 = A_0, \ \ B_1 = A_1 - T_1', \tag{18}$$

and

$$\begin{aligned} B_2 &= A_2 + [A_0, T_1] \\ B_3 &= A_3 + [A_1, T_1] + T_1 T_1' \\ B_4 &= A_4 + [A_2, T_1] - T_1 [A_0, T_1] \end{aligned} \tag{19}$$

with brackets denoting commutator of matrices. Note that according to our notation $T_1'(\varepsilon^{-1}t) = \varepsilon(\frac{\mathrm{d}}{\mathrm{d}t} T_1(\varepsilon^{-1}t))$, so that $T_1'(\varepsilon^{-1}t) = O(1)$. By setting

$$T_1 = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \tag{20}$$

we get

$$B_1 = A_1 - T_1' = 0,$$

as follows from (15) and (11). Substituting (20) into (19) we compute

$$B_2 = -\frac{1}{4}\begin{pmatrix} 0 & 2S \\ I & 0 \end{pmatrix}, \ \ B_3 = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}, \ \ B_4 = -\frac{1}{16}\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}; \tag{21}$$

summarizing, our equation becomes

$$\dot{z}_2 = (B_0 + \varepsilon^2 B_2(t) + \varepsilon^3 B_3 + \varepsilon^4 B_4(t) + O(\varepsilon^5))z_2. \tag{22}$$

5.1.2. Averaging of the ε^2-term. We now eliminate t from the $B_2(t)$ term in (22) by seeking the transformation

$$z_2 = (I + \varepsilon^3 T_2)z_3, \ \ T_2 = T_2(t/\varepsilon), \tag{23}$$

where $T_2(\tau)$ is a matrix function periodic in τ of period π. Substitution of (23) into (17) gives the new system

$$\dot{z}_3 = M_2 z_3$$

where

$$M_2 = (I + \varepsilon^3 T_2)^{-1} M_1 (I + \varepsilon^3 T_2) - (I + \varepsilon^3 T_2)^{-1} \varepsilon^2 T_2' \tag{24}$$

and M_1 is the coefficient of z_2 in (17). Note that we used the fact that $\dot{T}_2 = \varepsilon^{-1} T_2'$. Multiplying out (24) and collecting the like powers of ε we obtain

$$M_2 = A_0 + \varepsilon^2 (B_2 - T_2') + \varepsilon^3 \ (B_3 + [A_0, T_2]) + \varepsilon^4 B_4(t) + O(\varepsilon^5); \tag{25}$$

note that the ε^4-term was unaffected by the transformation. To kill the t-dependence in the ε^2-term we choose T_2 so as to turn $B_2 - T_2'$ into the average of B_2:

$$B_2 - T_2' = \overline{B_2} = -\frac{1}{4}\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \tag{26}$$

This condition, along with the requirement of periodicity, dictates the choice

$$T_2 = -\frac{1}{4}\begin{pmatrix} 0 & SJ \\ 0 & 0 \end{pmatrix}. \tag{27}$$

Substituting this into (25) yields

$$M_2 = A_0 + \varepsilon^2 \overline{B_2} + \varepsilon^3 B_3 + \varepsilon^4 B_4(t) + O(\varepsilon^5),$$

where we used the fact that $B_3 + [A_0, T_2] = B_3$, since T_2 commutes with A_0.

5.1.3. Reduction of the ε^4-term. Note that the cubic term turned out to be time-independent, and thus we need to average the quartic term. To that end we subject the system

$$\dot{z}_3 = M_2 z_3$$

to the transformation

$$z_3 = (I + \varepsilon^5 T_4) z_5$$

with the periodic matrix function T_4 chosen so as to kill time dependence in $B_4(t)$[4]. The matrix

$$M_4 = (I + \varepsilon^5 T_4)^{-1} M_2 (I + \varepsilon^5 T_4) - (I + \varepsilon^5 T_4)^{-1} \varepsilon^3 T_4'$$

of the transformed system differs from M_2 only in the terms starting with ε^4:

$$M_4 = M_2 - \varepsilon^4 T_4',$$

and thus we must choose T_4 so as to kill the time-dependence in the coefficient of ε^4:

$$T_4' = B_4(t) = -\frac{1}{16}\begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix},$$

or

$$T_4 = -\frac{1}{32}\begin{pmatrix} 0 & 0 \\ SJ & 0 \end{pmatrix}. \tag{28}$$

Denoting $z_5 = w$, we obtain the averaged system

$$\dot{w} = (A_0 + \varepsilon^2 \overline{B_2} + \varepsilon^3 B_3 + O(\varepsilon^5))\, w, \tag{29}$$

or, explicitly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\dfrac{\varepsilon^2}{4} I & \dfrac{\varepsilon^3}{4} J \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + O(\varepsilon^5)\begin{pmatrix} u \\ v \end{pmatrix}. \tag{30}$$

It follows that u satisfies

$$\ddot{u} - \frac{\varepsilon^3}{4} J \dot{u} + \frac{\varepsilon^2}{4} u = O(\varepsilon^4); \tag{31}$$

indeed, according to the first equation in (30)

$$\dot{u} = v + O(\varepsilon^5);$$

differentiating this by t gives

$$\ddot{u} = \dot{v} + O(\varepsilon^4)$$

– note the drop in the power of ε due to differentiation (recall that $\mathrm{d}/\mathrm{d}t = \varepsilon^{-1}\mathrm{d}/\mathrm{d}\tau$). Substituting $\dot{v}$ from the second equation in (30) results in (31).

It remains to find the explicit form for the averaging transformation

$$z_1 = (I + \varepsilon^2 T_1)(I + \varepsilon^3 T_2)(I + \varepsilon^5 T_4) w.$$

Expanding the product in the powers of ε, we write the transformation as

$$I + \varepsilon^2 T_1 + \varepsilon^3 T_2 + O(\varepsilon^5); \tag{32}$$

substituting the expressions for T_1 and T_2 (see (20) and (27)) and reading off the first component, we obtain (3), as claimed in the statement of theorem 1. $\diamond$

5.2. *Proof of theorem 2*

We start with the system (13) which was obtained from the original system by a contact transformation, and will show that the ε-term cannot be averaged by a contact transformation. In fact, it suffices to consider the transformation (16) since adding other powers of ε to $I + \varepsilon^2 T_1$

[4] We use the subscript 4 for consistency, noting that $T_3 = 0$, i.e. that the identity transformation is needed for the cubic terms.

will not affect the ε-term in the transformed system. Now T_1, which is uniquely determined by (20), gives the transformation with the matrix

$$I + \varepsilon^2 T_1 = \begin{pmatrix} I & 0 \\ 0 & I - \dfrac{1}{2}\varepsilon^2 S \end{pmatrix}. \tag{33}$$

But any map of contact type has the matrix of the form

$$\begin{pmatrix} \Phi & 0 \\ \Phi_t & \Phi \end{pmatrix}$$

where Φ is a matrix—this is just the linear version of (5) (we drop the t-variable since it is not transformed). And thus (33) is not of contact type. And adding other powers of ε to the matrix of the transformation (33) will not turn this transformation into one of contact type. This completes the proof of theorem 2. $\diamond$

6. Conclusion

We showed that the rapid rotation of the symmetric saddle potential creates a weak Lorentz-like, or a Coriolis-like force, in addition to an effective stabilizing potential—all in the inertial frame. As a result, the particle in the rotating saddle exhibits, in addition to oscillations caused by effective restoring force, a slow prograde precession caused by this pseudo-Coriolis effect. By finding a hodograph-like 'guiding center' transformation using the method of normal form, we found the effective equations of this precession that coincide with the equations of the Foucault's pendulum [36].

An interesting open question is to find a more abstract geometrical point of view (assuming one exists) from which the Coriolis-like force discussed here and geometric magnetism [37–39] are manifestations of the same effect.

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