Chapter 9

Index Theorems for Polynomial Pencils

9.1. Introduction

Spectral problems naturally arise in analyses of stability and decay rates of nonlinear waves, in stability analysis of numerical schemes, in integrable systems solved via inverse scattering method and in multiple other fields. The main interest is in presence of *unstable point spectrum* – points in the point spectrum of the particular operator with a positive real part that correspond to destabilizing modes. Over the last 40 years, counts of unstable point spectra and other related counts that we commonly refer to as *index theorems* have appeared across various distinct and unrelated fields due to their simple structure and importance for applications. Here, we briefly survey literature on index theory, point out its common graphical interpretation and derive its generalization to problems with operators with arbitrary structure of the kernel.

Linearized Hamiltonian systems: index theorems proved to be particularly useful in spectral stability theory of waves in Hamiltonian systems where we study the spectrum $\sigma(JL)$ of the non-self-adjoint problem

$$JLu = \nu u, \qquad J = -J^*,\ L = L^*, \qquad [9.1]$$

where J and L are operators acting on a Hilbert space X, L is the second variation Hessian of the underlying Hamiltonian, L^* denotes the adjoint operator of L and $u \in X$ [MEI 07, SWA 00]. Problem [9.1] appears in searching for exponentially growing or decaying solutions $v(x,t) = e^{\nu t}u(x)$ of the linearized system $v_t = JLv$ obtained by linearization of the Hamiltonian system around its equilibrium. Here,

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Chapter written by Richard KOLLÁR and Radomír BOSÁK.

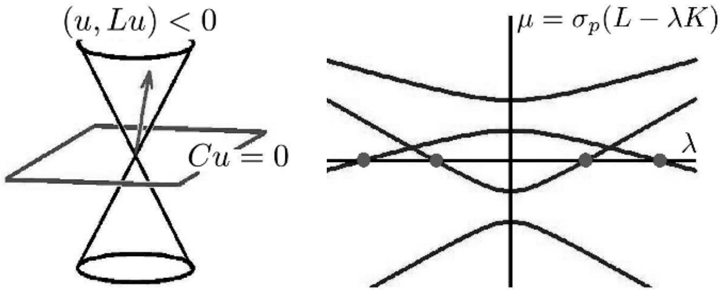
Nonlinear Physical Systems: Spectral Analysis, Stability and Bifurcations
Edited by Oleg N. Kirillov and Dmitry E. Pelinovsky

$v(x,t)$ represents an infinitesimal perturbation of the equilibrium that is said to be spectrally stable if [9.1] has no solution with $\mathrm{Re}\,\nu > 0$ for $u \in X$. Because of the natural symmetry of spectrum $\sigma(JL)$ (see [KOL 12b]), the spectral stability is equivalent to the confinement of $\sigma(JL)$ to the imaginary axis. Although positivity of the spectrum of L implies spectral stability, the operator L often has negative eigenvalues, and due to the symmetries of the system, also a non-trivial kernel allowing an instability in the system. However, the symmetries through the Noether theorem imply existence of conserved quantities (typically corresponding to physically quantities as mass, momentum, etc.). Their conservation restricts possible degrees of freedom in the system, and thus can prohibit instability in cases of indefinite L. The index theorems can be applied in such situations as they relate the number of negative real eigenvalues of L and the count of unstable spectra of JL. However, analogous index theorems proved to be useful in the quadratic operator pencils setting. The link between these two types of results is that for invertible J it is possible to reformulate [9.1] as a linear operator pencil. Therefore, both types of index theorems can be viewed as special cases of a general theory for operator pencils.

Two different ways to interpret the index theorems mathematically can be traced in the literature. Motivated by the work of Hestenes [HES 51] (see also [GRE 80]), Maddocks [MAD 88, MAD 95b] derived the dimension counts for finite-dimensional restricted quadratic forms and showed how the question of stability of an equilibrium of a Hamiltonian system reduces to a question whether a quadratic form is positive when restricted to a particular subspace of its domain. Such an approach was later used in [BEH 07, CHU 10, GRI 87b, HAR 08, KAP 04, PEL 05] and it is closely related to the theory of indefinite inner product spaces [BOG 74, IOH 56, LAN 82]. It provides a geometric visualization of the index theorems as counts of the dimensions of the intersection of the negative energy cone associated with the indefinite quadratic form with the subspace spanned by normal vectors (under the indefinite inner product) to hyperplanes tangential to surfaces of conserved quantities (Figure 9.1(a)).

However, here we focus on an alternative viewpoint of a different geometrical (graphical) nature [BIN 88, KOL 11, KOL 12b]. We interpret the index theorems as topological counts of curves of eigenvalues of operator pencils in a plane (Figure 9.1(b)). We believe that such an interpretation also provides an easier way for generalizations, besides the simpler visualization of the theory. Additionally, as we will show, it also yields reduced algebraic formulas for calculation of the indices of operators with complicated generalized kernels due to the fact that chains of root vectors generated by the graphical method carry extra information compared to (generic) chains of root vectors.

This work can be viewed as an extension of the theory developed by Kollár and Miller [KOL 12b] who laid down the groundwork for the analysis and derived special cases of theorems presented here. The present paper similarily to [KOL 12b] provides a list of the relevant literature on index theorems. However, these literature surveys are quite different. While [KOL 12b] exclusively focuses on the literature appearing in the field of stability of nonlinear waves, here we follow the idea of the BIRS workshop and bridge various different fields of theoretical and applied mathematics where the results appeared in parallel over the years. The lack of such survey gathering results from various fields served as a motivation for section 9.3. We hope that our work will contribute to increased communication, and thus faster transfer of results between various fields in the future.



(a) (b)

Figure 9.1. *Visualization of the index theory. a) Algebraic approach. The equilibrium is (spectrally) stable if and only if the normal vector (in the associated indefinite inner product space) to the plane $Cu = 0$ corresponding to the invariant (conserved quantity) of the system lies in the negative energy cone $\mathscr{C} = \{u \in X, (u, Lu) < 0\}$ [MAD 88]. b) Graphical approach. Eigenvalue branches $\mu = \mu(\lambda)$ (the point spectrum $\sigma_p(L - \lambda K)$) of the eigenvalue pencil $(L - \lambda K)u = \mu u$ are plotted versus λ. Purely imaginary eigenvalues ν of JL correspond to intercepts of $\mu(\lambda)$ with the axis $\mu = 0$ via $\lambda = i\nu$ (indicated by full circles). Their Krein signature is given by the sign of $\mu'(\lambda)$ [KOL 12b]*

9.2. Krein signature

Index theorems often refer to *Krein signature* of a characteristic value, a quantity that characterizes ability of the (stable) characteristic value to become unstable under a perturbation [KRE 50]. In linearized Hamiltonian systems, the Krein signature $\kappa_L(\lambda)$ of a characteristic value ν of *JL* captures the signature of the quadratic form $(\cdot, L\cdot)$ representing the linearized energy on the invariant subspace spanned by root spaces corresponding to $(\nu, -\overline{\nu})$ (see [MAC 87] for the geometric visualization). Krein signature is also referred to as *sign characteristics* [GOH 05] within the context of operator pencils and *symplectic signature* in Hamiltonian mechanics (see [KIR 09] for a detailed discussion of the terminology, literature survey, and extension of the results of [MAC 87]). If the signature of the quadratic form on the subspace is indefinite, the Krein signature is said to be *indefinite*, otherwise it is *definite (positive or negative)*. It is easy to see that the Krein signature of non-purely imaginary and non-semi-simple purely imaginary characteristic values is indefinite [IOH 56, PON 44]. However, the signature of any simple non-zero purely imaginary characteristic value of [9.1] is definite [KOL 12a].

If J is invertible, it is useful to define $K = (iJ)^{-1}$. Then, $Lu = i\nu Ku$ for a simple purely imaginary characteristic value ν of JL with the characteristic vector u and

$$\kappa_L(\nu) := \text{sign}(Lu, u) = \text{sign}\, i\nu(u, Ku).$$

Thus, the sign of (u, Ku) agrees with $\kappa_L(\nu)$ up to the sign of $i\nu$ and we can define

$$\kappa_K(\lambda) := -(u, Ku), \qquad \text{for } \lambda := i\nu, \quad Lu = \lambda Ku. \tag{9.2}$$

Since the definition [9.2] is closely related to its graphical analog, we will drop the index K in [9.2] in the remaining chapter where confusion will not arise. Due to the rotation $\lambda = i\nu$, the main interest lies in Krein signature of $\lambda \in \mathbb{R}$, so we limit ourselves to a definition of Krein signature of real characteristic values of [9.2].

In [KOL 12b], the detailed definitions of a characteristic value of an operator pencil, its geometric and algebraic multiplicities is given. The definitions of the maximal chain of root vectors and the canonical set of maximal chains can be also found in [GOH 05].

DEFINITION 9.1.– *Let J be an invertible skew-adjoint and L a self-adjoint operator on the Hilbert space X. Let λ_0 be a real characteristic value of iJL and let $\mathscr{U}$ be one of its maximal chains of root vectors. Furthermore, let $K = (iJ)^{-1}$ and let W be the (Hermitian) Gram matrix of the quadratic form $(\cdot, -K\cdot)$ on the span of $\mathscr{U}$. The number of positive (negative) eigenvalues of W is called the positive (negative) Krein index of $\mathscr{U}$ at λ_0 and is denoted $\kappa^+(\mathscr{U}, \lambda_0)$ ($\kappa^-(\mathscr{U}, \lambda_0)$). The sums of $\kappa^\pm(\mathscr{U}, \lambda_0)$ over the canonical set of maximal chains of root vectors $\mathscr{U}$ of λ_0 are called the positive and negative Krein indices of λ_0 and are denoted $\kappa^\pm(\lambda_0)$. Finally, $\kappa(\mathscr{U}, \lambda_0) := \kappa^+(\mathscr{U}, \lambda_0) - \kappa^-(\mathscr{U}, \lambda_0)$ is called the Krein signature of the maximal chain $\mathscr{U}$ for λ_0, and $\kappa(\lambda_0) := \kappa^+(\lambda_0) - \kappa^-(\lambda_0)$ is called the Krein signature of λ_0.*

In [GOH 05, KOL 12b], the proper analogous definition of Krein indices and Krein signature of a real characteristic value of a self-adjoint operator pencil of various types are given (Hermitian matrix pencils, compact perturbations of identity, holomorphic families of type (A) [KAT 76], etc.).

Now, we consider the spectrum of the operator pencil $\mathscr{L}(\lambda) := L - \lambda K$, i.e. the set of $\mu = \mu(\lambda)$ for which there exists $u \in X$ such that

$$\mathscr{L}(\lambda)u = \mu u. \qquad [9.3]$$

There is a natural one-to-one correspondence (including partial multiplicities) of the real point spectrum of *iJL* and the set of real characteristic values of $\mathscr{L}(\lambda)$, i.e. the set of $\lambda_0 \in \mathbb{R}$ such that $\mathscr{L}(\lambda_0)$ has a non-trivial kernel [GOH 05, KOL 12b, MAR 88]. Under suitable assumptions, the eigenvalues $\mu(\lambda)$ and eigenvectors $u(\lambda)$ can be chosen to be real analytic in λ [KOL 12b] and it is possible to define the graphical Krein signature for a self-adjoint operator pencil [KOL 12b, MAR 88].

DEFINITION 9.2.– *Let $\mathscr{L}(\lambda)$ be a self-adjoint operator pencil. Assume that $\mathscr{L}$ has an isolated real characteristic value λ_0 and there are real analytic eigenvalue branches $\mu(\lambda)$ of $\mathscr{L}(\lambda)$ such that eigenvalues of $\mathscr{L}(\lambda)$ for λ close to λ_0 are identical to $\mu(\lambda)$. Let $\mu = \mu(\lambda)$ be one of the branches with $\mu^{(n)}(\lambda_0) = 0$ for $n = 0, 1, \ldots, m-1$, and $\mu^{(m)}(\lambda_0) \neq 0$. Let $\eta(\mu) := \text{sign}(\mu^{(m)}(\lambda_0)) = \pm 1$. Then, the quantities*

$$\kappa_{\mathrm{g}}^{\pm}(\mu, \lambda_0) := \begin{cases} \frac{1}{2}m, & \text{for } m \text{ even}, \\ \frac{1}{2}(m \pm \eta(\mu)), & \text{for } m \text{ odd}, \end{cases} \qquad [9.4]$$

are called the positive and negative graphical Krein indices of the eigenvalue branch $\mu = \mu(\lambda)$ at λ_0. The sums of $\kappa_{\mathrm{g}}^{\pm}(\mu, \lambda_0)$ over all eigenvalue branches crossing at $(\lambda, \mu) = (\lambda_0, 0)$ are called the positive and negative graphical Krein indices of λ_0 and are denoted $\kappa_{\mathrm{g}}^{\pm}(\lambda_0)$. Finally, $\kappa_{\mathrm{g}}(\mu, \lambda_0) := \kappa_{\mathrm{g}}^{+}(\mu, \lambda_0) - \kappa_{\mathrm{g}}^{-}(\mu, \lambda_0)$ is called the graphical Krein signature of the eigenvalue branch $\mu = \mu(\lambda)$ vanishing at λ_0, and $\kappa_{\mathrm{g}}(\lambda_0) := \kappa_{\mathrm{g}}^{+}(\lambda_0) - \kappa_{\mathrm{g}}^{-}(\lambda_0)$ is called the graphical Krein signature of λ_0.

Definition 9.2 extends to general self-adjoint operator pencils as long as smooth eigenvalue and eigenvector branches exist in a neighborhood of an isolated characteristic value λ_0 [KOL 12b]. The fundamental relation between the Krein signature and the graphical Krein signature of a real λ_0 [BIN 88, KOL 12b, MAR 88] is given by

$$\kappa_{\mathrm{g}}(\lambda_0) = \kappa_K(\lambda_0) := \kappa(\lambda_0). \qquad [9.5]$$

The Krein signature $\kappa(\lambda_0)$ of a characteristic value λ_0 of $\mathscr{L} = \mathscr{L}(\lambda)$ then can be read off the graph of spectrum of $\mathscr{L}(\lambda)$ in the vicinity of $\lambda = \lambda_0$ and a maximal chain of root vectors of *iJL* at λ_0 can be generated by derivatives of the eigenfunction branch $u(\lambda)$ corresponding to the eigenvalue $\mu(\lambda)$ of [9.3] at $\lambda = \lambda_0$ [KOL 11, KOL 12b, MAR 88]. Relation [9.5] was rigorously established for Hermitian matrix pencils in [GOH 05] and for self-adjoint holomorphic operator pencils of type (A) in [KOL 12b].

9.3. Index theorems for linear pencils and linearized Hamiltonians

Let X and Y be separable Hilbert spaces and let A be a densely defined operator $D(A) \subset X \to Y$. We denote $\sigma_p(A)$ the point spectrum of A and $n_{\text{uns}}(A)$ denote the unstable index of A counting the number of points in $\sigma_p(A) \cap \{\text{Re}(z) > 0\}$. Furthermore, let $p(A)$, $z(A)$ and $n(A)$ be, respectively, the counts of positive, zero and negative real points in $\sigma_p(A)$ (counting multiplicity).

In 1972, Vakhitov and Kolokolov [VAK 73] studied stability of stationary (in an appropriate reference frame) solutions ϕ_ω of a nonlinear Schrödinger equation parameterized by angular velocity ω. Their linear stability is characterized by the spectrum of the eigenvalue problem [9.1]. The *Vakhitov–Kolokolov criterion* states that if $L_\pm$ are self-adjoint operators, L_+ is positive definite, L_- has exactly one negative eigenvalue,

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \tag{9.6}$$

then

$$n_{\text{uns}}(JL) = n(L) - n(dI/d\omega). \tag{9.7}$$

Here, $n(L) = 1$ and $I(\omega) = \int \phi_\omega^2 \, dx$ is the charge (momentum) of the stationary solution, i.e. $dI/d\omega$ is a scalar. The quantity $dI/d\omega$ can be related to the sign of the derivative $D'(\lambda)$ at $\lambda = 0$ of the Evans function [PEG 92] and to the quadratic form $(\cdot, L\cdot)$ evaluated at the first generalized characteristic vector of JL associated with the root vector ϕ_ω. Thus, the count of unstable point spectra of JL depends on the number related to the properties of the generalized kernel of JL given by $\text{gKer}(JL) := \text{span}_{k\geq 0} \text{Ker}\left((JL)^k\right)$.

Under the assumption of the full Hamiltonian symmetry $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$ of $\sigma_p(JL)$, it is straightforward to generalize [9.7] to the parity index theorem: $n_{\text{uns}}(JL) - (n(L) - n(D))$ is an even non-positive integer. Here, $n(D)$ is the count of negative eigenvalues of a matrix D related to $\text{gKer}(JL)$ (see theorem 9.1). This parity index theorem was proved in 1987 in celebrated papers [GRI 87a, GRI 87b] by Grillakis, Shatah, and Strauss who also proved the connection of the spectral stability to nonlinear stability for a wide class of problems (see also [SUL 99, Chapter 4] and [PEL 12] for the survey of the stability results in the context of nonlinear

Schrödinger and KdV equations). The generalization of the parity index theorem for the operators related to stability of dispersive waves was proved by Lin [LIN 08]. The parity index theorem also plays an important role in the theory of gyroscopic stabilization where it states that if the degree of instability (negative index) of the system is odd, the equilibrium point cannot be stabilized by gyroscopic forces. This fundamental theorem is often referred to as Thomson theorem [CHE 61, KOZ 05, THO 67], Thomson–Tait–Chetaev theorem [MER 97, Chapter 6] and Kelvin's theorem [KOZ 09] (Lord Kelvin's original name was William Thomson). See the works of Kozlov [KOZ 93, KOZ 10], Kozlov and Karapetyan [KOZ 05], and Chern [CHE 02] for further results, applications and references, and Kozlov [KOZ 09] for the topological implications of the theorem.

It is easy to identify $n_{\text{uns}}(JL) = k_r + 2k_c$, where k_r is the number of positive real points in $\sigma_p(JL)$ and k_c is the count of points in $\sigma_p(JL) \cap \{z \in \mathbb{C}, \text{Re}(z) > 0, \text{Im}(z) > 0\}$. Furthermore, denote

$$k_i^- = \sum_{\nu \in \sigma_p(JL) \cap i\mathbb{R}, i\nu < 0} \kappa_L^-(\nu) = \sum_{\lambda \in \sigma_p(iJL) \cap \mathbb{R}^-} \kappa_K^-(\lambda) .$$

The quantity k_i^- counts the total negative Krein index of points in $\sigma_p(JL) \cap i\mathbb{R}^+$. The final form of the index theorem for linearized Hamiltonians was proved in 2004 independently by Kapitula, Kevrekidis and Sandstede [KAP 04] and by Pelinovsky [PEL 05] (some assumptions of [PEL 05] were removed in [VOU 06]).

THEOREM 9.1.– [BIN 88, KAP 04, PEL 05]– Let J be an invertible skew-adjoint and L a self-adjoint operator acting on a Hilbert space X, with J^{-1} bounded on a subspace of X of a finite codimension, $n(L) < \infty$, and $\sigma(L) \cap \{x \in \mathbb{R}, x \leq 0\} \subset \sigma_p(L)$. Assume that the operators J and L satisfy (symmetry) assumptions that imply the full Hamiltonian symmetry of $\sigma_p(JL)$. Also assume that all Jordan chains corresponding to kernel of JL have length two. Let $\mathscr{V} = \text{gKer}(JL) \ominus \text{Ker}(L)$ and let D be the (symmetric) matrix of the quadratic form $(\cdot, L\cdot)$ restricted to $\mathscr{V}$. Then

$$n_{\text{uns}}(JL) = k_r + 2k_c = n(L) - n(D) - 2k_i^- . \quad [9.8]$$

Note that the count [9.8] was already established in 1988 by Binding and Browne [BIN 88, proposition 5.5] (although the $n(D)$ term is calculated a different way when $z(L) > 0$). They considered the case of L semi-bounded with compact resolvent and J 1-to-1 and used the standard perturbation theory combined with the graphical Krein signature referred to as *two parameter spectral theory*. Their short and simple argument is based on graphical inspection of eigenvalue branches (eigencurves) that can be interpreted as a homotopy in the parameter μ_0 from $L + \mu_0\mathbb{I}$ positive definite to L indefinite and counting of the eigenvalue branches' intersections with the axis

$\mu = 0$ (see also [BIN 96] where applications in Sturm–Liouville theory are studied). Furthermore, the claim and the proof of theorem 9.1 is to some extent implicitly present in the theory developed by Iohvidov [IOH 56], Langer [LAN 82] and also in Bognar [BOG 74, section XI.4].

For the operators arising in the spectral theory of Sturm–Liouville problems with indefinite weight, the index theorem [9.8] plays an important role. Although the upper bound of $n_{\text{uns}}(JL)$ is well understood [ZET 05, theorem 5.8.2], the exact count and dependence of its individual factors on the coefficients of the underlying differential equation poses an important open problem [ZET 05, problems IX–X, p. 300, problem 1, p. 124, comment (7), p. 128].

Kollár and Miller [KOL 12b] gave a short graphical proof of the index theorem [9.8] for Hermitian matrix pencils. We further generalize their results in section 9.4 and derive the generalization of a finite-dimensional version of theorem 9.1. Hârâgus and Kapitula [HAR 08] proved the analog of [9.8] for periodic Hamiltonian systems using the Floquet theory (Bloch wave decomposition) under technical assumptions related to the Keldysh theorem that guarantees completeness of the eigenvectors for [9.1]. Some of the technical assumptions of [HAR 08] were later removed in [DEC 10] (see also [BRO 11] for an alternative proof of [9.8] based on the integrable structure of the underlying problem). Recently, Kapitula and Stefanov [KAP 13] and Pelinovsky [PEL 12] removed the assumption of boundedness of J^{-1} and proved [9.8] for the case covering the Korteweg-de Vries-type problems with $J = \partial_x$ under the assumption $\dim(\text{Ker}L) = 1$ (see [PEL 12] for historical discussion of the stability results). Chugunova and Pelinovsky [CHU 10] studied the generalized eigenvalue problem $Lu = \lambda Ku$ using the theory of indefinite inner product spaces and particularly Pontryagin invariant subspace theorem and proved counts (inertia laws) analogous to [9.8]. Furthermore, they showed how [9.1] can be treated within that context and provided an alternative proof of [9.8].

In 1988, Jones [JON 88] and Grillakis [GRI 88] independently proved the index theorem bounding the number of unstable points in $\sigma_p(JL) \cap \mathbb{R}^+$ from below for the systems with the canonical form [9.6].

THEOREM 9.2.– [JON 88, GRI 88]– Let J and L have the canonical structure [9.6] with $L_\pm$ self-adjoint on a Hilbert space X, $\text{Ker}(L_+) \perp \text{Ker}(L_-)$ and let V denote the orthogonal complement of $\text{Ker}(L_+) \oplus \text{Ker}(L_-)$ in X with the orthogonal projection $P : Y \to V$. Then

$$n_{\text{uns}}(JL) \geq k_r \geq |n(PL_+P) - n(PL_-P)| . \qquad [9.9]$$

The proofs in [GRI 88] and in [JON 88] are significantly different, with the method of Grillakis [GRI 88] related to the graphical Krein signature. Note that theorem 9.2

does not rely on completeness of the root vectors of JL. Theorem 9.2 is frequently used to establish instability of various nonlinear waves, particularly in situations when the negative spectrum and the kernel of $L_{\pm}$ are explicitly known.

Kapitula and Promislow [KAP 12] reproved theorem 9.1 using the theory of [MAD 88] for constrained Hamiltonian systems and the Krein matrix theory, and reformulated [9.1], [9.6] by inverting the operator L^{+} reducing [9.1] to a generalized eigenvalue problem for which they proved [9.9]. They also proved a local count theorem analogous to theorem 9.4. Note that theorem 9.2 can also be easily obtained as a corollary of a general result of [KOL 12b] (see section 9.4) by using the same reformulation as in [KAP 12]. Both counts [9.8] and [9.9] were in a more general context also derived by Cuccagna *et al.* [CUC 05] in the setup, allowing the point spectrum to be embedded in the essential spectrum under some further technical assumptions. A lower bound for the number of real eigenvalues for Hermitian matrix pencils was derived by Lancaster and Tismenetsky [LAN 83] together with various other index theorems for perturbed Hermitian matrix pencils (including the upper bound for n_{uns}). Also, see Grillakis [GRI 90] for the analysis of the case $n(PL_{+}P) = n(PL_{-}P)$.

Quadratic eigenvalue pencils and their spectrum are a well-studied subject with a large number of applications (see [GOH 69, TIS 01] and references therein). Particular areas where index theorems naturally appear are Sturm–Liouville problems [BEH 09] and gyroscopic stabilization. Gyroscopic stabilization and stability of quadratic operator pencils in general are related to the point spectrum of the pencil $\mathscr{L}(\lambda) = \lambda^2 A + \lambda(D + iG) + K + iN$, where the coefficients A, D, G, K, N are self-adjoint operators (see [KIR 06, KOZ 10, MER 97] and references therein) under various additional conditions for the coefficients. A survey of all important results in this area is out of the scope of this chapter, and thus here we list only a few references. Fundamental results for quadratic operator pencils were obtained by Krein and Langer [KRE 78a, KRE 78b] and later extended by Adamyan and Pivovarchik [ADA 98] who also proved an index theorem similar to [9.8]. Results that can be expressed in a form of an index theorem were also obtained by Lancaster *et al.* [LAN 03, LAN 98] Wimmer [WIM 75] and Chern [CHE 02] (also see reference therein). Important index theorems for systems with dissipation $D > 0$ and partial dissipation $D \geq 0$ were proved in [KOZ 93, WIM 74, ZAJ 64]. Results of Zajac [ZAJ 64] that generalized the Thompson theorem for quadratic matrix pencils were later extended to operator setting by Pivovarchik in [PIV 92].

Within the field of stability of nonlinear waves, the index theorems for quadratic eigenvalue pencils are a fairly new subject. Chugunova and Pelinovsky [CHU 09] proved the count analogous to [9.8] for the quadratic Hermitian matrix pencils of the form $\lambda^2 \mathbb{I} + \lambda L + M$, where M has either zero- or one-dimensional kernel (under a further non-degeneracy condition) using the Pontryagin invariant subspace theorem. Their results were reproved and extended in [KOL 11, KOL 12b] (see example 9.1).

Bronski, Johnson, and Kapitula [BRO 12] proved a count similar to [9.8] for the quadratic operator pencils $\mathscr{L}(\lambda) = A + \lambda B + \lambda^2 C$, where A and C are self-adjoint and B is invertible skew-symmetric extending results of [PIV 07], [SHK 96], and [LYO 93].

Specific counts of eigenvalues for a particular class of Sturm–Liouville operators given by JL with $J = \text{sign}(x)$ and $L = -d^2/dx^2 + V(x)$ in $L^2(\mathbb{R})$ were obtained in [BEH 09, KAR 09]. Various types of index and eigenvalue localization theorems for definite and indefinite Sturm–Liouville problems, and particularly those that correspond to defective symmetric operators, can be found in [BEH 07, BEH 11] and in multiple reference therein (see also Binding and Volkmer [BIN 96] where non-real spectra of JL is studied with the use of graphical Krein signature). Further bounds particularly related to graphical Krein signature and eigenvalue branches $\mu(\lambda)$ (see section 9.4) were derived in [BIN 88]. The local count referred to as the *Krein oscillation theorem* was proved within the context of index theorems by Kapitula [KAP 10] using the Krein matrix theory. An infinitesimal version of the (local index) theorem 9.4 for matrices is proved in [GOH 05, theorem 12.6].

The theorem guaranteeing existence of a sequence of points in spectrum converging to zero for a general class of operator pencils with compact self-adjoint non-negative coefficients was proved in [KOL 11] by a simple homotopy argument as a generalization of the results of [GUR 04] (see also references therein). A homotopy argument was also used by Maddocks and Overton [MAD 95a] to prove the index theorem for dissipative perturbations of Hamiltonian systems. Index theorems within the context of isoperimetric calculus of variations were proved in [GRE 00]. Bronski and Johnson [BRO 09] derived an index theorem for the Faddeev-Takhtajan problem by an approach analogous to the work of Klaus and Shaw [KLA 02, KLA 03] on the Zakharov–Shabat system. Also, Kozlov and Karapetyan [KOZ 05] formulated the index theorem for finite-dimensional Hamiltonian systems that bounds the stable index of the system from below and related the result to gyroscopic stabilization.

To enclose the historical review of results on index theorems, let us point out that an unusually large part of the work mentioned within this section can be traced back to the University of Maryland at College Park, where many of the papers were written and many of the ideas were born. J.H. Maddocks, I. Gohberg, L. Greenberg, C.K.R.T. Jones, M. Grillakis, R.L. Pego and one of the authors of this chapter (R. Kollár) were among the others who were involved in the development of the theory.

9.4. Graphical interpretation of index theorems

In this section, we derive an index theory that encompasses theorems 9.1 and 9.2 and demonstrates their graphical nature. Although theorem 9.3 was derived in

[KOL 12b], our main results contained in theorems 9.4 and 9.5 generalize the theory developed in [KOL 12b]. The analysis is for simplicity performed for matrix pencils, although the results under specific assumptions can be generalized to infinitely dimensional setting [BIN 88] in a straightforward manner.

DEFINITION 9.3.– *Let $\mathscr{L} = \mathscr{L}(\lambda)$ be a Hermitian matrix pencil real analytic in λ and λ_0 its real characteristic value. Let $Z^{\downarrow}_{\lambda_0^-} = Z^{\downarrow}_{\lambda_0^-}(\mathscr{L})$ and $Z^{\downarrow}_{\lambda_0^+} = Z^{\downarrow}_{\lambda_0^+}(\mathscr{L})$ denote the number (counting multiplicity) of eigenvalue curves $\mu = \mu(\lambda)$ of $\mathscr{L}(\lambda)$ with $\mu(\lambda_0) = 0$ and $\mu(\lambda) < 0$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, respectively for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, for a sufficiently small $\varepsilon > 0$. Similarly, let $Z^{\downarrow}_{-\infty} = Z^{\downarrow}_{-\infty}(\mathscr{L})$ and $Z^{\downarrow}_{+\infty} = Z^{\downarrow}_{+\infty}(\mathscr{L})$ denote the number (counting multiplicity) of eigenvalue curves $\mu = \mu(\lambda)$ of $\mathscr{L}$ with $\mu(\lambda) < 0$ for $\lambda \in (-\infty, -K)$ and $\lambda \in (K, \infty)$, respectively, for a sufficiently large $K > 0$.*

The theory applies to real analytic Hermitian matrix pencils $\mathscr{L}(\lambda)$, i.e. real analytic $\mathscr{L} : \mathbb{R} \to \mathbb{C}^{n\times n}$ for which $\mathscr{L}(\lambda)$ is Hermitian for each $\lambda \in \mathbb{R}$, and thus generalizes the typical case of polynomial Hermitian matrix pencils.

THEOREM 9.3.– Let $\mathscr{L}(\lambda)$ be a real analytic Hermitian matrix pencil. Then

$$\left(Z^{\downarrow}_{-\infty} + Z^{\downarrow}_{+\infty}\right) - 2n(\mathscr{L}(0)) - \left(Z^{\downarrow}_{0^+} + Z^{\downarrow}_{0^-}\right) = -\sum_{\substack{\lambda\neq 0\\ \lambda\in\sigma(\mathscr{L})}} \operatorname{sign}(\lambda)\kappa(\lambda) \quad [9.10]$$

$$\left(Z^{\downarrow}_{-\infty} - Z^{\downarrow}_{+\infty}\right) + \left(Z^{\downarrow}_{0^+} - Z^{\downarrow}_{0^-}\right) = \sum_{\substack{\lambda\neq 0\\ \lambda\in\sigma(\mathscr{L})}} \kappa(\lambda). \quad [9.11]$$

PROOF.– Consider the (parameter-dependent) eigenvalue problem [9.3]. According to the perturbation theory [KAT 76], the eigenvalue and eigenvector curves $\mu(\lambda)$ and $u(\lambda)$ are analytic in λ. Furthermore, let $Q_\pm$ be the quadrants in the (λ, μ)-plane (see Figure 9.2). A simple count of curves entering and leaving $Q_\pm$ yields the counts

$$Z^{\downarrow}_{-\infty}(\mathscr{L}) - n(\mathscr{L}(0)) - Z^{\downarrow}_{0^-}(\mathscr{L}) - \sum_{\lambda<0,\lambda\in\sigma(\mathscr{L})} \kappa(\lambda) = 0, \quad [9.12]$$

$$Z^{\downarrow}_{+\infty}(\mathscr{L}) - n(\mathscr{L}(0)) - Z^{\downarrow}_{0^+}(\mathscr{L}) + \sum_{\lambda>0,\lambda\in\sigma(\mathscr{L})} \kappa(\lambda) = 0, \quad [9.13]$$

Then, the sum and the difference of [9.12] and [9.13] give [9.10] and [9.11].

A local version of the index theorem can be proved analogously.

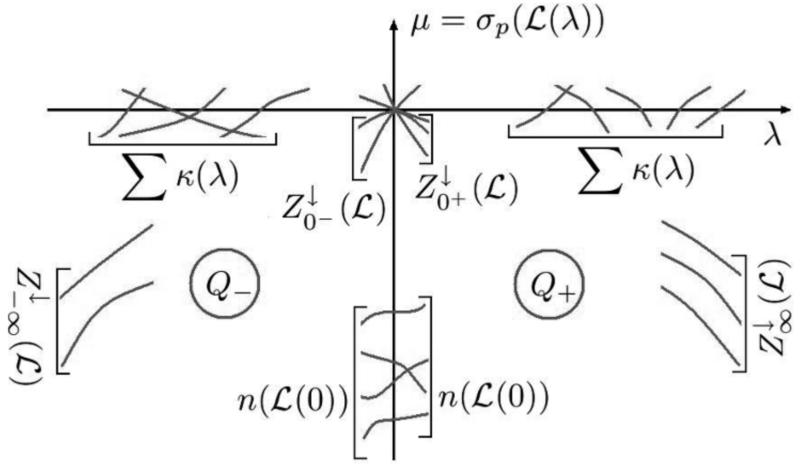


Figure 9.2. *A schematic plot of the proof of theorem 9.3. The point spectrum $\sigma_p(\mathscr{L}(\lambda))$ is organized in eigenvalue branches $\mu = \mu(\lambda)$*

THEOREM 9.4.– Let $\mathscr{L}(\lambda)$ be a real analytic Hermitian matrix pencil. Then, the following local index theorem holds for any real λ_1, λ_2 with $\lambda_1 < \lambda_2$:

$$n\big(\mathscr{L}(\lambda_1)\big) - n\big(\mathscr{L}(\lambda_2)\big) + Z^{\downarrow}_{\lambda_1^+} - Z^{\downarrow}_{\lambda_2^-} = \sum_{\substack{\lambda_1 < \lambda < \lambda_2 \\ \lambda \in \sigma(\mathscr{L})}} \kappa(\lambda)\,. \qquad [9.14]$$

It is easy to see that

$$Z^{\downarrow}_{\lambda_0^-} - Z^{\downarrow}_{\lambda_0^+} = Z^{\uparrow}_{\lambda_0^+} - Z^{\uparrow}_{\lambda_0^-} = \kappa(\lambda_0)\,, \qquad [9.15]$$

since the eigenvalue branches vanishing at λ_0 at even order do not contribute to the right-hand side of [9.15]. Then

$$Z^{\downarrow}_{\lambda_0^-} + Z^{\downarrow}_{\lambda_0^+} = \kappa(\lambda_0) + 2Z^{\downarrow}_{\lambda_0^+}\,, \qquad Z^{\downarrow}_{\lambda_0^-} + Z^{\uparrow}_{\lambda_0^-} = Z^{\downarrow}_{\lambda_0^+} + Z^{\uparrow}_{\lambda_0^+} = z(\mathscr{L}(\lambda_0))\,. \quad [9.16]$$

Moreover, if the matrix pencil $\mathscr{L}(\lambda)$ has an extra structure, then the terms $Z^{\downarrow}_{\pm\infty}(\mathscr{L})$ can be determined by the perturbation theory. Let

$$\mathscr{L}(\lambda) = L(\lambda) - g(\lambda)\mathbb{I}, \qquad L(\lambda) = \sum_{k=0}^{p} \lambda^k L_k\,, \qquad g(\lambda) = \sum_{k=0}^{q} \lambda^k g_k. \qquad [9.17]$$

Here, $L_0,\dots,L_p$ are complex Hermitian matrices $n \times n$ and $g_0,\dots g_q$ are real constants. There is a freedom of choice in an inclusion of identity multipliers in $g(\lambda)$ and $L(\lambda)$ but the index theorems only depend on the leading order term of $\mathscr{L}(\lambda)$ and only differences of $g(\lambda)$ and $L(\lambda)$ are relevant. Therefore, we ignore such an ambiguity in [9.17] and assume $\lambda^p L_p \neq \lambda^q g_q \mathbb{I}$. Since $\sigma(L(\lambda)) \approx \lambda^p \sigma(L_p)$ and $g(\lambda) \approx \lambda^q g_q$ for $|\lambda| \to \infty$, the values of terms $Z^{\downarrow}_{\pm\infty}$ in [9.10] and [9.11] are determined by the leading order coefficients of $g(\lambda)$ and $L(\lambda)$.

THEOREM 9.5.– Let $\mathscr{L}(\lambda)$ be a real analytic Hermitian matrix pencil of the form [9.17] and let L_p be invertible. Then, the values of the counts $Z^{\downarrow}_{-\infty}(\mathscr{L})$ and $Z^{\downarrow}_{+\infty}(\mathscr{L})$ appearing in theorem 9.3 are given by the values in the table below (depending on the properties of $L(\lambda)$ and $g(\lambda)$):

p,q	g_q	$Z^{\downarrow}_{-\infty}(\mathscr{L})$	$Z^{\downarrow}_{+\infty}(\mathscr{L})$
$q>p$, q even	$g_q>0$	n	n
$q>p$, q even	$g_q<0$	0	0
$q>p$, q odd	$g_q>0$	0	n
$q>p$, q odd	$g_q<0$	n	0
$q<p$		$n\big((-1)^p L_p\big)$	$n(L_p)$
$q=p$		$n\big((-1)^p (L_p - q_p\mathbb{I})\big)$	$n(L_p - q_p\mathbb{I})$

By setting $g(\lambda)=0$, i.e. $q=0$, and thus $q<p$ for a non-trivial pencil $\mathscr{L}(\lambda)$, in theorems 9.3 and 9.5, we can recover generalizations of theorems 9.1 and 9.2 in a finite dimensional case [KOL 12b]. Next, we illustrate how specific counts for quadratic eigenvalue pencils can be derived from theorems 9.3 and 9.5.

EXAMPLE 9.1.– Consider the quadratic Hermitian matrix pencil $\mathscr{L}(\lambda)$, $\mathscr{L}:\mathbb{R}\to\mathbb{C}^{n\times n}$

$$\mathscr{L}(\lambda) = M + \lambda K + \lambda^2 \mathbb{I}, \qquad M^* = M, \quad K^* = K. \tag{9.18}$$

Then, $L(\lambda) = M + \lambda K$, $p=1$, and $g(\lambda) = -\lambda^2$, $q=2$, and $g_q=-1$ in [9.17]. Therefore, $Z^{\downarrow}_{-\infty} = Z^{\downarrow}_{+\infty} = 0$ by theorem 9.5. Theorem 9.3 then gives

$$2n(M) + (Z^{\downarrow}_{0^+} + Z^{\downarrow}_{0^-}) = \sum_{\lambda\in\sigma(\mathscr{L})} \operatorname{sign}(\lambda)\kappa(\lambda), \quad Z^{\downarrow}_{0^+} - Z^{\downarrow}_{0^-} = \sum_{\substack{\lambda\neq 0\\ \lambda\in\sigma(\mathscr{L})}} \kappa(\lambda).$$

The symmetry $(\lambda,\overline{\lambda})$ of spectrum of $\mathscr{L}$ implies

$$2n = z(\mathscr{L}) + n_r(\mathscr{L}) + 2n_c(\mathscr{L}) + n_i(\mathscr{L}), \tag{9.19}$$

where $n_r(\mathscr{L})$, $n_i(\mathscr{L})$ and $n_r(\mathscr{L})$ are, respectively, the numbers of real, purely imaginary and non-real non-purely imaginary characteristic values of $\mathscr{L}(\lambda)$, $n_i(\mathscr{L})$ is even. Also

$$\begin{aligned} n_r(\mathscr{L}) &= \sum_{\lambda\in\sigma(\mathscr{L}),\lambda>0} \left[\kappa^+(\lambda)+\kappa^-(\lambda)\right] + \sum_{\lambda\in\sigma(\mathscr{L}),\lambda<0} \left[\kappa^+(\lambda)+\kappa^-(\lambda)\right] \\ &= \sum_{\lambda\in\sigma(\mathscr{L})} \operatorname{sign}(\lambda)\kappa(\lambda) + 2\sum_{\lambda\in\sigma(\mathscr{L}),\lambda>0} \kappa^-(\lambda) + 2\sum_{\lambda\in\sigma(\mathscr{L}),\lambda<0} \kappa^+(\lambda)\,. \end{aligned}$$

Finally, we denote

$$n_r^-(\mathscr{L}) := 2\sum_{\lambda\in\sigma(\mathscr{L}),\lambda>0} \kappa^-(\lambda)\,, \qquad n_r^+(\mathscr{L}) := 2\sum_{\lambda\in\sigma(\mathscr{L}),\lambda>0} \kappa^+(\lambda)\,, \tag{9.20}$$

motivated by the case of simple real characteristic values of $\mathscr{L}$ where n_r^- (respectively n_r^+) counts the number of characteristic values of $\mathscr{L}$ with the negative (respectively positive) Krein signature. Then, [9.19] can be rewritten as

$$2n(M) + (Z_{0+}^{\downarrow} + Z_{0-}^{\downarrow}) = n_r(\mathscr{L}) - n_r^-(\mathscr{L}) - n_r^+(\mathscr{L})\,. \tag{9.21}$$

The symmetric case: if the eigenvalue problem $\mathscr{L}(\lambda)u = \mu u$ has an additional symmetry $(\lambda,\mu) \to (-\lambda,\mu)$, the index theorem can be further simplified. Clearly, $Z_{0+}^{\downarrow} = Z_{0-}^{\downarrow}$, $\kappa^-(\lambda) = \kappa^+(\lambda)$, i.e. $n_r^-(\mathscr{L}) = n_r^+(\mathscr{L})$, and $n_r^+(\mathscr{L}) + n_r^-(\mathscr{L}) = n_r(\mathscr{L})$. Furthermore, all the numbers $z(\mathscr{L}), n_r(\mathscr{L}), n_c(\mathscr{L})$ are even. A difference of [9.19] and [9.21] yields

$$\left[n - \frac{z(\mathscr{L})}{2}\right] - \left[n(M) + Z_{0+}^{\downarrow}\right] = n_c(\mathscr{L}) + \frac{n_i(\mathscr{L})}{2} + n_r^-(\mathscr{L}) \tag{9.22}$$

The equation complementary to [9.22] with respect to [9.19] is

$$\left[n - \frac{z(\mathscr{L})}{2}\right] + \left[n(M) + Z_{0+}^{\downarrow}\right] = n_c(\mathscr{L}) + \frac{n_i(\mathscr{L})}{2} + n_r^+(\mathscr{L}) \tag{9.23}$$

In the special case of a real Hermitian M and purely imaginary Hermitian L and under the assumption $\operatorname{Ker} M \subset \operatorname{Ker} L$, the counts [9.22]–[9.23] correspond to index theorems derived in [CHU 09] and [KOL 11].

9.4.1. *Algebraic calculation of* $Z^{\downarrow}$ *and* $Z^{\uparrow}$

The counts [9.10], [9.11], and [9.14] contain terms $Z^{\downarrow}$, $Z^{\uparrow}$ that have a simple graphical interpretation. However, it is more traditional to express them in an algebraic form that we derive in this section. Theorem 9.6 generalizes the relation between $Z^{\downarrow}$ and $n(D)$ in theorem 9.1 (see [KOL 12b] for details) that holds in the case of all Jordan blocks of the eigenvalue 0 of JL of length two (implying $\dim \mathrm{gKer}(JL) = 2\dim \mathrm{Ker}(L)$) to the case of the generalized kernel of $\mathscr{L}(\lambda_0)$ of an arbitrary structure. First, we formulate the general assumption that guarantees the required smoothness of the eigenvalue and eigenvector branches at the characteristic value of an operator pencil.

ASSUMPTION 9.1.– Let $\mathscr{L} = \mathscr{L}(\lambda)$ be a real analytic self-adjoint operator pencil acting on a Hilbert space X and let λ_0 be its characteristic value of a finite multiplicity. Let $\mathscr{U}_0 = \mathrm{Ker}(\mathscr{L}(\lambda_0))$ with $\dim \mathscr{U}_0 = k$, and let $\varepsilon > 0$, $\delta > 0$ are fixed. Assume that for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ the part of the spectrum $\sigma(\mathscr{L}(\lambda)) \cap (-\delta, \delta)$ of $\mathscr{L}(\lambda)$ consists of eigenvalues organized in C^∞ eigenvalue branches $\mu_j(\lambda)$, $\mu_j(\lambda_0) = 0$, $\mu_j : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to (-\delta, \delta)$, $j = 1, \ldots, k$, and that the associated eigenvector branches $u_j(\lambda)$, $u_j : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $1 \leq j \leq k$, are also C^∞.

Note that assumption 9.1 is satisfied for Hermitian matrix pencils for and $\varepsilon > 0$ and $\delta > 0$ [KAT 76]. Define for every $m \geq 0$ the sets

$$\begin{aligned} K_m^+ &:= \{\mu_i(\lambda);\ 1 \leq i \leq k, \mu_i^{(s)}(\lambda_0) = 0,\ 0 \leq s \leq m-1,\ \mu^{(m)}(\lambda_0) > 0\}, \\ K_m^- &:= \{\mu_i(\lambda);\ 1 \leq i \leq k, \mu_i^{(s)}(\lambda_0) = 0,\ 0 \leq s \leq m-1,\ \mu^{(m)}(\lambda_0) < 0\}, \\ K_m^0 &:= \{\mu_i(\lambda);\ 1 \leq i \leq k, \mu_i^{(s)}(\lambda_0) = 0,\ 0 \leq s \leq m\}, \end{aligned} \qquad [9.24]$$

The sets K_m^+, K_m^- and K_m^0 are disjoint for any $m \geq 0$ and

$$K_m^0 = K_{m+1}^- \cup K_{m+1}^+ \cup K_{m+1}^0, \qquad m \geq 0. \qquad [9.25]$$

For a characteristic value λ_0 of $\mathscr{L}(\lambda)$ of a finite multiplicity; $K_m^0 = \emptyset$ for m large enough. Then

$$Z^{\downarrow}_{\lambda_0^+} = \Big|\bigcup_{m=1}^{\infty} K_m^-\Big| = \sum_{m=1}^{\infty} |K_m^-|,\ Z^{\downarrow}_{\lambda_0^-} = \Big|\bigcup_{m=1}^{\infty} K_{2m-1}^+ \cup K_{2m}^-\Big| = \sum_{m=1}^{\infty} |K_{2m-1}^+| + |K_{2m}^-|.$$

Also, observe that $n(\mathscr{L}(\lambda_0)) = |K_0^-|$ and that the algebraic multiplicity of the characteristic value λ_0 of $\mathscr{L}$ is given by $\sum_{m=1}^{\infty} m\,(|K_m^+| + |K_m^-|)$.

We claim that $|K_m^\pm|$, $m \geq 0$, can be calculated as the number of positive (negative) eigenvalues of a specific matrix defined in theorem 9.6. Particularly, for m odd, $|K_m^\pm|$ counts the number of maximal chains of root vectors of $\mathscr{L}$ at λ_0 with positive (negative) Krein index. Therefore, the Krein index $\kappa(\mathscr{U},\lambda_0)$ can be calculated by two different ways, either from the (algebraic) definition or by using the graphical Krein signature.

EXAMPLE 9.2.– Let $\mathscr{L}(\lambda) = M + \lambda L + \lambda^2 \mathbb{I}$ be a quadratic Hermitian matrix pencil and let $\mathscr{U} = \{u^{[0]}, u^{[1]}, u^{[2]}\}$ be a maximal chain of root vectors of $\mathscr{L}(\lambda)$ at a characteristic value $\lambda_0 = 0$. According to the definition of the Krein indices analogous to definition 9.1 (see [KOL 12b] for details), the indices $\kappa^\pm(\mathscr{U},\lambda_0)$ count the number of positive and negative eigenvalues of the Gram matrix W

$$W_{ij} = (u^{[i-1]}, Lu^{[j-1]}) + (u^{[i-2]}, u^{[j-1]}) + (u^{[i-1]}, u^{[j-2]}), \;\; for \;\; i,j = 1,2,3,$$

where we formally set $u^{[-1]} = 0$. The characteristic polynomial $f(\lambda) = \det(W - \lambda\mathbb{I})$ is a cubic polynomial with negative leading order coefficient and three real roots, either two of them positive and one negative or one of them positive and two negative. Thus

$$\kappa(\mathscr{U},\lambda_0) = -\operatorname{sign} f(0) = -\operatorname{sign}(\det W). \qquad [9.26]$$

EXAMPLE 9.3.– Consider the quadratic pencil in example 9.2 and its characteristic value $\lambda_0 = 0$ with a maximal chain of root vectors $\mathscr{U} = \{u^{[0]}, u^{[1]}, u^{[2]}\}$. According to the theory [KOL 12b] (see [GOH 05] for the matrix case), there exist eigenvalue and eigenvector branches $\mu(\lambda)$, $u(\lambda)$ of [9.3] such that $\mu(0) = \mu'(0) = \mu''(0) = 0$ and $u(0) = u^{[0]}$. Also $\kappa(\mathscr{U},0) = \operatorname{sign}\mu'''(0) \neq 0$. For notational ease, we denote $\mathscr{L} = \mathscr{L}(0)$, $u = u(0)$, $\mu = \mu(0)$, with the analogous notation for the derivatives: $\mu' = \mu'(0)$, $u' = u'(0)$, etc. We normalize $(u(0),u(0)) = 1$, differentiate [9.3] three times and take the scalar product with u to obtain

$$\left(u, \frac{\mathscr{L}'''}{3!}u\right) + \left(u, \frac{\mathscr{L}''}{2!}u'\right) + \left(u, \mathscr{L}'\frac{u''}{2!}\right) + \left(u, \mathscr{L}\frac{u'''}{3!}\right) = \frac{\mu'''}{3!}. \qquad [9.27]$$

Since $\mathscr{L}$ is Hermitian, the last term on the left-hand side of [9.27] vanishes. Also, differentiation of [9.3] implies $\mathscr{L}u' + \mathscr{L}'u = 0$ and $\mathscr{L}u'' + 2\mathscr{L}'u' + \mathscr{L}''u = 0$. The operator $\mathscr{L} = \mathscr{L}(0)$ is not invertible but $\mathscr{L} + \Pi$, where Π is the orthogonal projection $X \to \operatorname{Ker}\mathscr{L}(0)$, is. We denote $\tilde{\mathscr{L}}^{-1} = -(\mathscr{L} + \Pi)^{-1}$ and note that $\tilde{\mathscr{L}}^{-1}\Pi = -\Pi$. Then

$$u' = \tilde{\mathscr{L}}^{-1}\mathscr{L}'u + \Pi u', \quad \text{and} \quad \frac{u''}{2!} = \tilde{\mathscr{L}}^{-1}L'u' + \tilde{\mathscr{L}}^{-1}\frac{\mathscr{L}''}{2!}u + \Pi\frac{u''}{2!}. \qquad [9.28]$$

Simple algebra (see theorem 9.6 for the derivation in the general case) reduces [9.27] to

$$\kappa(\mathscr{U},\lambda_0) = \operatorname{sign}\mu'''(0) = \operatorname{sign}\left[(u,\Lambda_3 u) - (\Pi u', L\Pi u')\right], \qquad [9.29]$$

where Λ_3 is defined in [9.32]. In the case of the quadratic pencil

$$\Lambda_3 = (M+\Pi)^{-1}L + L(M+\Pi)^{-1} + L(M+\Pi)^{-1}L(M+\Pi)^{-1}L.$$

NOTE.– Formula [9.29] has important consequences. It contains only $u = u^{[0]}$ and $u' = u'(0)$, i.e. it does not require knowledge of the whole maximal chain $\mathscr{U}$, contrary to [9.26]. Also the term $(\Pi u', L\Pi u')$ is, generally, non-vanishing and since $\Pi u' \in \operatorname{Ker}\mathscr{L}$, its value is not directly encoded in a chain $\mathscr{U}$ as the generalized eigenvectors are determined uniquely only up to a multiple of $u^{[0]}$. It means that $u'(0)$ is not just an arbitrary generalized root vector to $u^{[0]}$ but it captures a piece of extra information $\Pi u'$ that is not, in general, contained in $u^{[1]}$. Thus, the chain of root vectors $(u(0), u'(0), u''(0)/2)$ is exceptional, which will also be confirmed in a general case of an arbitrary multiplicity. A similar calculation in the case of a quadratic matrix pencil can be found in [BOS 12].

As it was illustrated in example 9.3, the graphical approach requires a proper definition of the inverse of the operator $\mathscr{L}(\lambda_0)$. If $\mathscr{L}(\lambda_0)$ is Fredholm and self-adjoint, then $\operatorname{Ker}\mathscr{L}(\lambda_0) \perp \operatorname{Ran}\mathscr{L}(\lambda_0)$. The operator $\mathscr{L}(\lambda_0)$ acts on the Hilbert space $X = \operatorname{Ker}\mathscr{L}(\lambda_0) \oplus \operatorname{Ran}\mathscr{L}(\lambda_0)$. If $(v_1, v_2) \in \operatorname{Ker}\mathscr{L}(\lambda_0) \oplus \operatorname{Ran}\mathscr{L}(\lambda_0)$, then $\mathscr{L}(\lambda_0)(v_1, v_2) = (0, \mathscr{L}(\lambda_0)v_2)$. The operator $\mathscr{L}(\lambda_0)$ is 1-to-1 on $\operatorname{Ran}\mathscr{L}(\lambda_0)$, and thus the operator $\mathscr{L}(\lambda_0) + \Pi$ is invertible as $(\mathscr{L}(\lambda_0) + \Pi)(v_1, v_2) = (v_1, \mathscr{L}(\lambda_0)v_2)$.

DEFINITION 9.4.– Let $\mathscr{L} = \mathscr{L}(\lambda)$ *be an operator pencil acting on a Hilbert space* X *with a characteristic value* λ_0 *of a finite multiplicity, and let* $\mathscr{L}(\lambda_0)$ *have the Fredholm index zero. Let* Π *be an orthogonal projection* $X \to \operatorname{Ker}\mathscr{L}(\lambda_0)$. *Then, we define*

$$\tilde{\mathscr{L}}^{-1} := -(\mathscr{L}(\lambda_0) + \Pi)^{-1}. \qquad [9.30]$$

Clearly, $\mathscr{L}\Pi = 0$ and $\tilde{\mathscr{L}}^{-1}\Pi = -\Pi$. Also, denote $D := d/d\lambda$.

NOTATION.– We introduce the following notation. Let V be a linear subspace of X of the dimension k with the orthonormal basis $\{v_1, \ldots, v_k\}$ and let S be a self-adjoint operator acting on S. First, let V^*SV denote the matrix of quadratic form $(\cdot, S\cdot)$ acting on V, i.e. the matrix (v_i, Sv_j), $1 \le i, j \le k$. Then, $\widehat{\operatorname{Ker}}(V^*SV)$ denotes the $k \times m$ matrix with its columns given by the m column vectors that form the basis of the kernel of V^*SV. Finally, let $W := V\widehat{\operatorname{Ker}}(V^*SV)$ define the subspace of X of the dimension m

spanned by vectors obtained by the multiplication of the vector $(v_1, \ldots, v_k) \in X^k$ by the m columns of the matrix $\widehat{\mathrm{Ker}}(V^*SV)$.

THEOREM 9.6.– Let $\mathscr{L} = \mathscr{L}(\lambda)$ be an operator pencil on a Hilbert space X with a characteristic value λ_0 satisfying assumption 9.1, and let $\mathscr{L}(\lambda_0)$ be of Fredholm index zero. Let $K_m^{\pm}$, $m \geq 0$, be defined in [9.24] and let $U_0 = \mathrm{Ker}\,\mathscr{L}(\lambda_0) \subset X$. We define recursively

$$U_{m+1} := U_m\, \widehat{\mathrm{Ker}}\,(U_m^* H_{m+1}\, U_m)\,, \qquad [9.31]$$

The operator H_m, $m \geq 1$ is defined as $H_m := \Lambda_m + \mathscr{D}_m$. Here

$$\Lambda_m \;:=\; \sum_{|\alpha|=m} \frac{\mathscr{L}^{(\alpha_1)}(\lambda_0)}{\alpha_1!}\,\tilde{\mathscr{L}}^{-1}\,\frac{\mathscr{L}^{(\alpha_2)}(\lambda_0)}{\alpha_2!}\,\tilde{\mathscr{L}}^{-1}\cdots\cdot\tilde{\mathscr{L}}^{-1}\,\frac{\mathscr{L}^{(\alpha_s)}(\lambda_0)}{\alpha_s!}, \qquad [9.32]$$

$$\mathscr{D}_m \;:=\; \sum_{|\alpha|=m} D^{\alpha_1}\Pi\Lambda_{\alpha_2}\Pi D^{\alpha_3}\,, \qquad [9.33]$$

where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_s)$ has positive integer entries and its norm is calculated as $|\alpha| = \sum_{i=1}^{s} \alpha_i$. Then, for $m \geq 1$

$$|K_m^+| = p\,(U_{m-1}^*\, H_m\, U_{m-1})\,,\; |K_m^-| = n\,(U_{m-1}^*\, H_m\, U_{m-1})\,,\; U_m^* H_{m+1}\, U_{m+1} = 0\,.$$

PROOF.– We prove theorem 9.6 by mathematical induction for $m \geq 1$. Without loss of generality, we set $\lambda_0 = 0$.

First, let $m = 1$. Let us fix $\mu_i \in K_0^0$. Then

$$(\mathscr{L}(\lambda) - \mu_i(\lambda))u_i(\lambda) = 0\,. \qquad [9.34]$$

Differentiation of [9.34] with respect to λ at $\lambda = \lambda_0$, where $\mu_i(0) = 0$ and a scalar product with u_j such that $\mu_j \in K_0^0$ yields

$$\big(u_j, (\mathscr{L}' - \mu_i')u_i\big) + (u_j, \mathscr{L}u_i') = 0\,. \qquad [9.35]$$

where for notational ease, we drop the argument of $\mathscr{L}$, μ and u and their derivatives. The second term in [9.35] vanishes as $(u_j, \mathscr{L}u_i') = (\mathscr{L}u_j, u_i') = 0$. Therefore

$$\big(u_j, \mathscr{L}'u_i\big) = \mu_i'\,(u_j, u_i) = \mu_i'\delta_{ij}\,, \qquad [9.36]$$

with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. As a result, the matrix $(u_j, \mathscr{L}' u_i)$, $u_i, u_j \in \mathrm{Ker}\,\mathscr{L}(0)$, $j = 1, \ldots, k$, is diagonal with its eigenvalues on a diagonal. The number of its positive, negative and zero eigenvalues is independent of a choice of basis of $\mathrm{Ker}\,\mathscr{L}(0)$ and it is determined by the signature of the quadratic form $(\cdot, \mathscr{L}' \cdot)$ on $\mathrm{Ker}\,\mathscr{L}(0)$. Since $\Lambda_1 = \mathscr{L}'(0)$ and $\mathscr{D}_1 = 0$, we derived

$$|K_1^+| = p\,(U_0^* H_1\, U_0)\,, \qquad |K_m^-| = n\,(U_0^* H_1\, U_0)\,.$$

Let $h \in \mathrm{Ker}(U_0^* H_1\, U_0)$. Then, $U_0^* H_1\, U_0 h = 0$, and if $v \in U_0\,\mathrm{Ker}(U_0^* H_1\, U_0) = U_1$ also $U_0^* H_1\, U_1 = 0$.

Now assume that the statement of the theorem holds for all j, $j \leq m$. Let $\mu_i \in K_{m-1}^0$. Differentiation of [9.34] m times in λ at $\lambda = 0$ together with $\mu_i(0) = \cdots = \mu_i^{(m-1)}(0) = 0$ gives

$$\sum_{j=0}^{m} \frac{\mathscr{L}^{(m-j)}}{(m-j)!}\,\frac{u_i^{(j)}}{j!} = \frac{\mu_i^{(m)}}{m!} u_i\,. \qquad [9.37]$$

Also,

$$\sum_{j=0}^{s} \frac{\mathscr{L}^{(s-j)}}{(s-j)!}\,\frac{u_i^{(j)}}{j!} = 0 \qquad [9.38]$$

for $s = 1, \ldots, m-1$. Adding the term $\Pi u_i^{(s)}/s!$ to both sides of [9.38] and inverting the operator $(\mathscr{L} + \Pi)$ yields an expression for $u_i^{(s)}$. Now, we rewrite [9.37] as

$$\frac{\mathscr{L}^{(m)}}{(m)!} u_i + \sum_{j=1}^{m} \frac{\mathscr{L}^{(m-j)}}{(m-j)!}\,\frac{u_i^{(j)}}{j!} = \frac{\mu_i^{(m)}}{m!} u_i\,. \qquad [9.39]$$

and express recursively each term $u_i^{(s)}/s!$, $1 \leq s < m$ that does not contain the projection operator Π until all the terms in the sum on the right-hand side contain either u_i or a projection operator Π. The total number of derivatives in each term in the expanded sum in [9.39] is invariant and equal to $(m-j) + j = m$. The derivatives in each term are partitioned into derivatives of multiple copies of $\mathscr{L}$ and one copy of

u_i in such a way that such possible partitions of m derivatives appear exactly once and thus equation [9.39] reduces to

$$\Lambda_m u_i - \sum_{s=1}^{m-1} \Lambda_{m-s} \Pi \frac{u_i^{(s)}}{s!} + \mathscr{L} \frac{u_i^{(m)}}{m!} = \frac{\mu_i^{(m)}}{m!} u_i \,. \qquad [9.40]$$

Taking scalar product of [9.40] with u_j such that $\mu_j \in K_m^0$ yields

$$(u_j, \Lambda_m u_i) - \sum_{s=1}^{m-1} (u_j, \Lambda_{m-s} \Pi \frac{u_i^{(s)}}{s!}) + (u_j, \mathscr{L} \frac{u_i^{(m)}}{m!}) = \frac{\mu_i^{(m)}}{m!} \delta_{ij} \,. \qquad [9.41]$$

Since $\mu_j \in K_m^0$, the root vector u_j can be also expressed for any $p < m$ as (compare to [9.40])

$$\Lambda_p u_j - \sum_{s=1}^{p-1} \Lambda_{p-s} \Pi \frac{u_j^{(s)}}{s!} + \mathscr{L} \frac{u_j^{(p)}}{p!} = 0 \,. \qquad [9.42]$$

Then, each individual term in the summand in the second term on the left-hand side of [9.41] can be reduced to

$$\begin{aligned} (u_j, \Lambda_{m-s} \Pi \frac{u_i^{(s)}}{s!}) &= (\Lambda_{m-s} u_j, \Pi \frac{u_i^{(s)}}{s!}) \\ &= \sum_{r=1}^{m-s-1} (\Lambda_{m-s-r} \Pi \frac{u_j^{(r)}}{r!}, \Pi \frac{u_i^{(s)}}{s!}) - (\mathscr{L} \frac{u_j^{(p)}}{p!}, \Pi \frac{u_i^{(s)}}{s!}) \\ &= \sum_{r=1}^{m-s-1} (\Lambda_{m-s-r} \Pi \frac{u_j^{(r)}}{r!}, \Pi \frac{u_i^{(s)}}{s!}) \end{aligned}$$

Since $(u_j, \mathscr{L} u_i^{(m)}/m!) = 0$, the expression in [9.41] can be rewritten as

$$(u_j, H_m u_i) = (u_j, \Lambda_m u_i) + (u_j, \mathscr{D}_m u_i) = \frac{\mu_i^{(m)}}{m!} \delta_{ij} \,. \qquad [9.43]$$

Therefore, the matrix $U_m^* H_m U_m$ is diagonal and

$$|K_{m+1}^+| = p\,(U_m^* H_m U_m)\,, \; |K_{m+1}^-| = n\,(U_m^* H_m U_m)\,, \; |K_{m+1}^0| = z\,(U_m^* H_m U_m)\,.$$

Multiplication of [9.43] by $h \in \mathrm{Ker}(U_m^* H_m U_m)$ gives $U_m^* H_m U_m h = 0$ that implies $U_m^* H_m U_{m+1} = 0$.

NOTE.– According to [9.33] we have $\mathscr{D}_1 = \mathscr{D}_2 = 0$. Also, there are two terms $z(\mathscr{L})$ and $Z_{0+}^{\downarrow}$ on the left-hand side of [9.23] that are connected with the properties of the (generalized) kernel of $\mathscr{L}$. It is easy to see that under an assumption $\mathrm{Ker}(M) \subset \mathrm{Ker}(L)$, we have $z(\mathscr{L}) = 2\dim(\mathrm{Ker}(M))$ that leads to the simplified expression in [9.8] (see [KOL 11] for details).

9.5. Conclusions

We presented a unifying view of the index theorems frequently used across various fields. Furthermore, we demonstrated a special property of the chain of root vectors generated by the graphical method that allowed us to derive formulas for the number of eigenvalue curves of the eigenvalue problem $\mathscr{L}(\lambda)u = \mu u$ entering the lower half-plane of the plane (μ, λ) through the characteristic value λ_0 of $\mathscr{L}(\lambda)$ that did not require knowledge of the full chain of the root vectors. Both these results demonstrate the extraordinary beauty and power of the graphical approach. Let us conclude with a quotation from [BIN 96]: *"Eigencurves* (produced by the graphical approach) *seem to provide a very useful tool in a variety of circumstances, and their theory and applications are quite underdeveloped"*, a statement that certainly remains true even today.

9.6. Acknowledgments

R. Kollár would like to thank the European Commission for the financial support via the Marie Curie International Reintegration Grant 239429.

9.7. Bibliography

[ADA 98] ADAMYAN V., PIVOVARCHIK V., *On The Spectra of Some Classes of Quadratic Operator Pencils*, Birkhäuser Verlag, Basel, pp. 23–36, 1998.

[BEH 07] BEHRNDT J., TRUNK C., "On the negative squares of indefinite Sturm–Liouville operators", *Journal of Differential Equations*, vol. 238, pp. 491–519, 2007.

[BEH 09] BEHRNDT J., KATATBEH Q., TRUNK C., "Non-real eigenvalues of singular indefinite Sturm–Liouville operators", *Proceedings of the American Mathematical Society*, vol. 137, no. 11, pp. 3797–3806, 2009.

[BEH 11] BEHRNDT J., MÖWS R., TRUNK C., "Eigenvalue estimates for singular left-definite Sturm–Liouville operators", *Journal of Spectral Theory*, vol. 1, no. 3, pp. 327–347, 2011.

[BIN 88] BINDING P., BROWNE P.J., "Applications of two parameter spectral theory to symmetric generalized eigenvalue problems", *Applicable Analysis*, vol. 29, pp. 107–142, 1988.

[BIN 96] BINDING P., VOLKMER H., "Eigencurves of two-parameter Sturm–Liouville equations", *SIAM Review*, vol. 38, no. 1, pp. 27–48, 1996.

[BOG 74] BOGNAR J., *Indefinite Inner Product Spaces*, Springer, New York, 1974.

[BOS 12] BOSÁK R., Counting the Krein signature by planar curves, Bachelor Thesis, Comenius University, Bratislava, 2012.

[BRO 09] BRONSKI J.C., JOHNSON M.A., "Krein signatures for the Faddeev–Takhtajan eigenvalue problem", *Communications in Mathematical Physics*, vol. 288, pp. 821–846, 2009.

[BRO 11] BRONSKI J.C., JOHNSON M.A., KAPITULA T., "An index theorem for the stability of periodic traveling waves of KdV type", *Proceedings of the Royal Society of Edinburgh: Section A*, vol. 141, pp. 1141–1173, 2011.

[BRO 12] BRONSKI J.C., JOHNSON M.A., KAPITULA T., "An instability index theory for quadratic pencils and applications", 2012. *arXiv*, 1207, 3764.

[CHE 61] CHETAYEV N.G., *The Stability of Motion*, Pergamon Press, Oxford, 1961.

[CHE 02] CHERN S.J., "Stability theory for linear dissipative Hamiltonian systems", *Linear Algebra and its Applications*, vol. 357, pp. 143–162, 2002.

[CHU 09] CHUGUNOVA M., PELINOVSKY D., "On quadratic eigenvalue problems arising in stability of discrete vortices", *Linear Algebra and its Applications*, vol. 431, pp. 962–973, 2009.

[CHU 10] CHUGUNOVA M., PELINOVSKY D., "Count of eigenvalues in the generalized eigenvalue problem", *Journal of Mathematical Physics*, vol. 51, p. 052901, 2010.

[CUC 05] CUCCAGNA S., PELINOVSKY D., VOUGALTER V., "Spectra of positive and negative energies in the linearized NLS problem", *Communications on Pure and Applied Mathematics*, vol. 58, pp. 1–29, 2005.

[DEC 10] DECONINCK B., KAPITULA T., "On the orbital (in)stability of spatially periodic stationary solutions of generalized Korteweg-de Vries equations", preprint, 2010.

[GOH 69] GOHBERG I.C., KREIN M.G., *Introduction to the Theory of Linear Nonselfadjoint Operators*, vol. 18, Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1969.

[GOH 05] GOHBERG I., LANCASTER P., RODMAN L., *Indefinite Linear Algebra and Applications*, Birkhäuser Verlag, Basel, 2005.

[GRE 80] GREGORY J., *Quadratic Form Theory and Differential Equations*, Academic Press, New York, 1980.

[GRE 00] GREENBERG L., MADDOCKS J.H., ROGERS K.A., "The bordered operator and the index of a constrained critical point", *Mathematische Nachrichten*, vol. 219, pp. 109–124, 2000.

[GRI 87a] GRILLAKIS M., SHATAH J., STRAUSS W.A., "Stability theory of solitary waves in the presence of symmetry II", *Journal of Functional Analysis*, vol. 94, pp. 308–348, 1987.

[GRI 87b] GRILLAKIS M., SHATAH J., STRAUSS W.A., "Stability theory of solitary waves in the presence of symmetry I", *Journal of Functional Analysis*, vol. 74, pp. 160–197, 1987.

[GRI 88] GRILLAKIS M., "Linearized instability for nonlinear Schrödinger and Klein–Gordon equations", *Communications on Pure and Applied Mathematics*, vol. 46, pp. 747–774, 1988.

[GRI 90] GRILLAKIS M., "Analysis of the linearization around a critical point of an infinite dimensional hamiltonian system", *Communications on Pure and Applied Mathematics*, vol. 43, pp. 299–333, 1990.

[GUR 04] GURSKI K.F., KOLLÁR R., PEGO R.L., "Slow damping of internal waves in a stably stratified fluid", *Proceedings of the Royal Society A*, vol. 460, pp. 977–994, 2004.

[HAR 08] HĂRĂGUŞM., KAPITULA T., "On the spectra of periodic waves for infinite-dimensional Hamiltonian systems", *Physica D*, vol. 237, pp. 2649–2671, 2008.

[HES 51] HESTENES M.R., "Applications of the theory of quadratic forms in Hilbert space in the calculus of variations", *Pacific Journal of Mathematics*, vol. l, pp. 525–581, 1951.

[IOH 56] IOHVIDOV I.S., KREIN M.G., "Spectral theory of operators in spaces with indefinite metric I", *Tr. Mosk. Mat. Obsc*, vol. 5, pp. 367–432, 1956. Also *American Mathematical Society Translations*, vol. 2, pp. 105–175, 1960.

[JON 88] JONES C.K.R., "Instability of standing waves for non-linear Schrödinger-type equations", *Ergodic Theory and Dynamical Systems*, vol. 8, pp. 119–138, 1988.

[KAP 04] KAPITULA T., KEVREKIDIS P., SANDSTEDE B., "Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems", *Physica D*, vol. 195, pp. 263–282, 2004. Also addendum, *Physica D*, vol. 201, pp. 199–201, 2004.

[KAP 10] KAPITULA T., "The Krein signature, Krein eigenvalues, and the Krein oscillation Theorem", *Indiana University Mathematics Journal*, vol. 59, pp. 1245–1276, 2010.

[KAP 12] KAPITULA T., PROMISLOW K., "Stability indices for constrained self-adjoint operators", *Proceedings of the Royal Society*, vol. 140, pp. 865–880, 2012.

[KAP 13] KAPITULA T., STEFANOV A., "A Hamiltonian-Krein (Instability) index theory for solitary waves to KdV-like eigenvalue problems", *Studies in Applied Mathematics*, 2013.

[KAR 09] KARABASH I.M., KOSTENKO A.S., MALAMUD M.M., "The similarity problem for j-nonnegative Sturm–Liouville operators", *Journal of Differential Equations*, vol. 246, pp. 964–997, 2009.

[KAT 76] KATO T., *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

[KIR 06] KIRILLOV O.N., "Gyroscopic stabilization of non-conservative systems", *Physics Letters A*, vol. 359, pp. 204–210, 2006.

[KIR 09] KIRILLOV O.N., "Campbell diagrams of weakly anisotropic flexible rotors", *Proceedings of the Royal Society A*, vol. 465, pp. 2703–2723, 2009.

[KLA 02] KLAUS M., SHAW J.K., "Purely imaginary eigenvalues of Zakharov–Shabat systems", *Physical Review E*, vol. 65, no. 3, 036607, 2002.

[KLA 03] KLAUS M., SHAW J.K., "On the eigenvalues of Zakharov–Shabat systems", *SIAM Journal on Mathematical Analysis*, vol. 34, no. 4, pp. 759–774, 2003.

[KOL 11] KOLLÁR R., "Homotopy method for nonlinear eigenvalue pencils with applications", *SIAM Journal on Mathematical Analysis*, vol. 43, pp. 612–633, 2011.

[KOL 12a] KOLLÁR R., PEGO R.L., "Spectral stability of vortices in two-dimensional Bose–Einstein condensates via the Evans function and Krein signature", *Appl. Math. Res. Express*, vol. 2012, pp. 1–46, 2012.

[KOL 12b] KOLLÁR R., MILLER P.D., "Graphical Krein signature theory and Evans–Krein functions", *arXiv*, 1209.3185, 2012.

[KOZ 93] KOZLOV V.V., "On the degree of instability", *Prikladnaya Matematika i Mekhanika*, vol. 57, no. 5, pp. 14–19, 1993. Also *Journal of Applied Mathematics and Mechanics*, vol. 57, no. 5, pp. 771–776, 1993.

[KOZ 09] KOZLOV V.V., "Kelvin's Instability Theorem: topological meaning and generalizations", *Doklady Akademii nuck*, vo. 424, no. 2, pp. 161–164, 2009. Also *Doklady Mathematics*, vol. 79, pp. 25–28, 2009.

[KOZ 10] KOZLOV V.V., "Remarks on the degree of instability", *Prikl. Mat. Mekh.*, vol. 74, no. 1, pp. 18–21, 2010. Also *Journal of Applied Mathematics and Mechanics.*, vol. 74, pp. 10–12, 2010.

[KOZ 05] KOZLOV V.V., KARAPETYAN A.A. "On the stability degree", *Diff. Uravneniya*, vol. 41, no. 2, pp. 186–192, 2005. Also *Differential Equations*, vol. 41, pp. 195–201, 2005.

[KRE 50] KREIN M.G., "A generalization of some investigations of A. M. Lyapunov on linear differential equations with periodic coefficients", *Doklady Akademii nauk SSSR*, vol. 73, pp. 445–448, 1950.

[KRE 78a] KREIN M.G., LANGER H., "On some mathematical principles in the linear theory of damped oscillations of continua I", *Integral Equations and Operator Theory*, vol. 1, pp. 364–399, 1978.

[KRE 78b] KREIN M.G., LANGER H., "On some mathematical principles in the linear theory of damped oscillations of continua II", *Integral Equations and Operator Theory*, vol. 1, pp. 539–566, 1978.

[LAN 82] LANGER H., *Spectral Functions of Definitizable Operators in Krein Spaces*, vol. 948, Lecture Notes in Mathematics 948, Springer-Verlag, New York, pp. 1–46, 1982.

[LAN 83] LANCASTER P., TISMENETSKY M., "Inertia characteristics of self-adjoint matrix polynomials" *Linear Algebra and its Applications*, vol. 52/53, pp. 479–496, 1983.

[LAN 98] LANCASTER P., ZIZLER P., "On the stability of gyroscopic systems", *Journal of Applied Mechanics*, vol. 65, pp. 519–522, 1998.

[LAN 03] LANCASTER P., MARKUS A.S., ZHOU F., "A wider class of stable gyroscopic systems", *Linear Algebra and its Applications*, vol. 370, pp. 257–267, 2003.

[LIN 08] LIN Z., "Instability of nonlinear dispersive solitary waves", *Journal of Functional Analysis*, vol. 255, pp. 1191–1224, 2008.

[LYO 93] LYONG N., "The spectral properties of a quadratic pencil of operators", *Russian Mathematical Surveys*, vol. 48, pp. 180–182, 1993.

[MAC 87] MACKAY R.S., *Stability of Equilibria of Hamiltonian Systems*, Hilger, Bristol, pp. 137–153, 1987.

[MAD 88] MADDOCKS J.H., "Restricted quadratic forms and their application to bifurcation and stability in constrained variational principles", *SIAM J. Math. Anal.*, vol. 16, pp. 47–68, 1988. Also Errata *SIAM Journal on Mathematical Analysis*, vol. 19, pp. 1256–1257, 1988.

[MAD 95a] MADDOCKS J.H., OVERTON M.L., "Stability theory for dissipatively perturbed Hamiltonian systems", *Communications on Pure and Applied Mathematics*, vol. 48, pp. 583–610, 1995.

[MAD 95b] MADDOCKS J.H., SACHS R.L., *Constrained Variational Principles and Stability in Hamiltonian Systems*, IMA Volumes in Mathematics and Its Applications, Minneapolis, MN, pp. 231–264, 1995.

[MAR 88] MARKUS A.S., *Introduction to the Spectral Theory of Polynomial Operator Pencils*, vol. 71, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1988.

[MEI 07] MEISS J.D., *Differential Dynamical Systems*, SIAM, PA, 2007.

[MER 97] MERKIN D.R., *Introduction to the Theory of Stability*, Springer-Verlag, New York, 1997.

[PEG 92] PEGO R.L., WEINSTEIN M.I., "Eigenvalues, and instabilities of solitary waves", *Philosophical Transactions of the Royal Society A*, vol. 340, pp. 47–94, 1992.

[PEL 05] PELINOVSKY D.E., "Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations", *Proceedings of the Royal Society*, vol. 461, pp. 783–812, 2007.

[PEL 12] PELINOVSKY D.E., Spectral Analysis, Stability, and Bifurcation in Modern Nonlinear Physical Systems, Banff, Canada, 4–9 November 2012.

[PIV 92] PIVOVARCHIK V.N., "On the total algebraic multiplicity of the spectrum in the right half-plane for a class of quadratic operator pencils", *St. Petersburg Mathematical Journal*, vol. 3, no. 2, pp. 447–454, 1992.

[PIV 07] PIVOVARCHIK V., "On spectra of a certain class of quadratic operator pencils with one-dimensional linear part", *Ukrainian Mathematical Journal*, vol. 59, pp. 766–781, 2007.

[PON 44] PONTRYAGIN L.S., "Hermitian operators in spaces with indefinite metric", *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, vol. 8, pp. 243–280, 1944.

[SHK 96] SHKALIKOV A., *Operator Pencils Arising in Elasticity and Hydrodynamics: the Instability Index Formula*, vol. 87, Operator Theory Advances and Applications, Birkhäuser Verlag, Basel, pp. 358–385, 1996.

[SUL 99] SULEM C., SULEM P.-L., *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*, Springer-Verlag, New York, 1999.

[SWA 00] SWATERS G.E., *Introduction to Hamiltonian Fluid Dynamics and Stability Theory*, Chapman & Hall/CRC, Boca Raton, 2000.

[THO 67] THOMSON W., TAIT P., *Treatise on Natural Philosophy*, Clarendon Press, Oxford, 1867.

[TIS 01] TISSEUR F., MEERBERGEN K., "The quadratic eigenvalue problem", *SIAM Review*, vol. 43, pp. 235–286, 2001.

[VAK 73] VAKHITOV N.G., KOLOKOLOV A.A., "Stationary solutions of the wave equation in a medium with nonlinearity saturation", *Izv. Vyssh. Uchebn. Zaved. Radiofiz.*, vol. 16, pp. 1020–1028, 1973. Also *Radiophys. Quantum Electron.*, vol. 16, pp. 783–789, 1973.

[VOU 06] VOUGALTER V., PELINOVSKY D., "Eigenvalues of zero energy in the linearized NLS problem", *Journal of Mathematical Physics*, vol. 47, p. 062701, 2006.

[WIM 74] WIMMER H.K., "Inertia theorems for matrices, controllability and linear vibrations", *Linear Algebra and its Applications*, vol. 8, no. 4, pp. 337–343, 1974.

[WIM 75] WIMMER H.K., "Remarks on inertia theorems for matrices", *Czechoslovak Mathematical*, vol. 25, no. 4, pp. 556–561, 1975.

[ZAJ 64] ZAJAC E.E., "The Kelvin–Tait–Chetaev theorem and extensions", *Journal of the Aeronautical Sciences*, vol. 11, no. 2, pp. 46–49, 1964.

[ZET 05] ZETTL A., *Sturm–Liouville Theory*, vol. 121, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005.