Chapter 8

Determining the Stability Domain of Perturbed Four-Dimensional Systems in 1:1 Resonance

We consider the ordinary differential equation $\dot{x} = F_\mu(x)$ with a stationary point at $x = 0$ for all μ, where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^p$ and $n = 4$. The linear part of the vector field at $x = 0$, $A : \mathbb{R}^p \to \mathrm{gl}(n, \mathbb{R}) : \mu \mapsto A(\mu) = DF_\mu(0)$, is in 1:1 resonance at $\mu = 0$ with imaginary eigenvalues. We are interested in the position of the parameter family $A(\mu)$ with respect to the stability domain $\mathscr{S}$ where $A(\mu)$ has all its eigenvalues in the left half of the complex plane. Upon varying the parameters μ along a path that transversally crosses the boundary $\partial\mathscr{S}$ of the stability domain, the stationary point changes stability, generically in a Hopf bifurcation. But at particular points, we find Hopf-Hopf bifurcations, nilpotent 1:1 resonances and a semi-simple 1:1 resonance at $A(0)$. We find explicit criteria for stability expressed in the parameters of the family $A(\mu)$ and apply them to a number of physical problems.

8.1. Introduction

8.1.1. *Physical motivation*

Many physical and engineering models are rotationally symmetric. A semi-simple 1:1 resonance (when two eigenfrequencies coalesce into a double eigenfrequency having two linearly independent eigenmodes) occurs more frequently in the presence of symmetry than in generic systems. Its unfolding due to symmetry-breaking perturbations is a reason for many important instabilities. For example, the rotating polygon instability of swirling free surface flow [TOP 13],

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magnetohydrodynamics dynamo action in the axisymmetric von Kármán-like flow subject to non-axisymmetric velocity perturbations [GEI 12], superharmonic two-dimensional instabilities of Stokes waves [MAC 86b] as well as the mass and stiffness instabilities in rotor dynamics [MOT 98, NAG 98], to name a few.

In linear Hamiltonian systems whether the unfolding of the semi-simple 1:1 resonance is destabilizing or stability preserving is determined by the symplectic [WIL 36] or Krein sign [MOR 98, KRE 83] of the eigenvalues merging into the double eigenvalues [MAC 86a, KIR 13b]. In cases where the double eigenvalue has definite Krein sign, the unfolding is a three-dimensional hypercone giving *avoided crossings* or *veering* [MAC 12, MOT 98, VID 05] of eigenvalue branches on a generic one-dimensional path in parameter space [MAC 86a]. When the Krein sign is indefinite or mixed, the three-dimensional MacKay's hypercone gives rings of both negative and positive real parts of the perturbed eigenvalues – *bubbles of instability* – in many one-parameter slices [MAC 86a]. Note that the Krein sign of a vibration mode (related to the sign of the mode's energy [LAM 08, STU 58]) determines also stabilization or destabilization under dissipative perturbations. Destabilization of negative-energy modes of a Hamiltonian system by the Rayleigh dissipation is a classical example of the dissipation-induced instability [BLO 94, KRE 06, KRE 07, MAC 91, MAD 95] formalized by the Kelvin–Tait–Chetaev theorem [KIR 13b, LAN 13].

In wave propagation problems, double semi-simple eigenvalues are known since Hamilton's discovery of conical refraction [GAR 89]. Von Neumann and Wigner [VON 29] and Teller [TEL 37] calculated the codimension of double semi-simple eigenvalues in the families of real symmetric and Hermitian matrices and established that the eigenvalues as functions of parameters have conical singularities at the doublets. Non-Hermitian perturbation of eigenvalues of Hermitian matrices initiated first in numerical analysis [KAH 75] has found numerous applications in physics in recent years. Non-Hermitian unfoldings of conical eigenvalue surfaces with the apexes at the double semi-simple eigenvalues play an important role in crystal optics [BER 03, KIR 05], rotor dynamics [KIR 13b], acoustics [SHU 00], open quantum systems [KEC 03] and stability of granular materials [AN 92, KIR 13b]. Particularly, it was observed that the non-Hermitian perturbation can transform the conical eigenvalue surface to a new surface known as the *conical wedge of Wallis* [KIR 13b] or the *double coffee filter* [KEC 03] with two "Whitney umbrella" singularities corresponding to double eigenvalues with a Jordan block [KIR 05].

Behavior of eigenvalues and eigenvectors in the vicinity of such branch points have many important interpretations in modern physics [BER 04, HEI 12, ELE 13]. In particular, in the vicinity of the branch points in the parameter space, the eigenvalues dramatically change their direction in the complex plane. For this reason, in stability problems, the branch points are considered as precursors to flutter instability, for example, in electrical engineering [DOB 01] and fluid dynamics

[JON 88]. Imaginary double eigenvalues with a Jordan block (nilpotent 1:1 resonance) are responsible for the linear Hamiltonian and reversible Hopf bifurcation that is connected to the linear Hopf bifurcation in dissipative systems by means of the "Whitney umbrella" singularity [BOT 56, HOV 95, LAN 03, SEY 03, KIR 13b, KIR 10, VAN 90]. The latter fact is a reason for the structural instability caused by small dissipation and known in mechanics as the Ziegler–Bottema destabilization paradox [BOT 55, KIR 13b, ZEI 53] taking place, for example, in rotor dynamics [CRA 70, CRA 95, HOV 95] and fluid mechanics [BRI 07, KRE 07, SWA 10, KIR 13a.]

In an analogy with the problems of non-Hermitian physics [BER 04, HEI 12, ELE 13], nilpotent 1:1 resonances can originate in the unfolding of the 1:1 semi-simple resonance [HOV 10]. In the following, we present a detailed study of this process in the case of general four-dimensional systems.

8.1.2. *Setting*

Suppose that the ordinary differential equation

$$\dot{x} = F_\mu(x) \tag{8.1}$$

has a stationary point at $x = 0$ for all μ. In this equation, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^p$ and $F_\mu(x)\frac{\partial}{\partial x}$ is a vector field on $\mathbb{R}^n$ smoothly depending on x and μ. Furthermore, suppose that the linear part of the vector field at $x = 0$, $A : \mathbb{R}^p \to \mathrm{gl}(n,\mathbb{R}) : \mu \mapsto A(\mu) = DF_\mu(0)$, is in 1:1 resonance at $\mu = 0$ with imaginary eigenvalues. This means that $A(0)$ is similar to

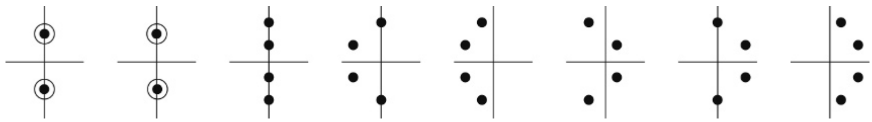
$$L = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{8.2}$$

Since $A(0)$ is only marginally stable, it is natural to ask for which values of μ $A(\mu)$ is stable so that we can draw conclusions about the stability of the stationary point $x = 0$ of equation [8.1]. In other words, we are interested in the position of the parameter family $A(\mu)$ with respect to the *stability domain*

$$\mathscr{S} = \{A \in \mathrm{gl}(n,\mathbb{R}) \mid \text{if } \lambda \text{ is an eigenvalue of } A, \text{ then } \mathrm{Re}(\lambda) \leq 0\} \tag{8.3}$$

for $n = 4$. It is clear from this definition that $A(0)$ is an element of the *boundary* $\partial\mathscr{S}$ of the stability domain. Upon varying the parameters μ along a path that transversally crosses $\partial\mathscr{S}$, the stationary point changes stability, generically in a Hopf bifurcation. But at particular points, we find Hopf-Hopf bifurcations and nilpotent 1:1 resonances.

The latter are as yet only known for Hamiltonian and reversible systems, see [VAN 85] and [VAN 94], however, also see [VAN 90] for general systems. At $A(0)$, we find a semi-simple 1:1 resonance, which is an unknown bifurcation even for Hamiltonian or reversible systems. In Figure 8.1, we show the possible *eigenvalue configurations* near $A(0)$, (see [HOV 10] for a definition).



$\beta\beta$ $\quad\beta^2\quad$ $\beta_1\beta_2$ $\quad\beta\gamma_-\quad$ $\gamma_{-1}\gamma_{-2}$ $\quad\gamma_-\gamma_+\quad$ $\beta\gamma_+$ $\quad\gamma_{+1}\gamma_{+2}$

Figure 8.1. *Eigenvalue configurations near $A(0)$. $\beta\beta$ is the eigenvalue configuration of $A(0)$. On $\partial\mathscr{S}$, we have $\beta\beta$, β^2, $\beta_1\beta_2$ or $\beta\gamma_-$; on $\mathscr{S}$, we only have $\gamma_{-1}\gamma_{-2}$ and elsewhere we have $\gamma_-\gamma_+$, $\beta\gamma_+$ or $\gamma_{+1}\gamma_{+2}$. Notation for eigenvalue configurations: α means real, β means imaginary and γ means complex eigenvalue, a sign denotes a negative real part, a number distinguishes different eigenvalues*

The stability question may be delicate near singularities of the boundary $\partial\mathscr{S}$ and, in particular, $A(0)$ is located at such a singular point. In [HOV 10], we address the question of singularities of $\partial\mathscr{S}$, in general, in a neighborhood in $\mathrm{gl}(4,\mathbb{R})$ of L. In that article, we concentrate on the *universal unfolding* of L without returning to the original deformation $A(\mu)$ of $A(0)$. Here, we wish to find explicit criteria for stability expressed in the parameters of the family $A(\mu)$, which is not necessarily a universal unfolding.

8.1.3. *Main question and examples*

The main aim of this chapter is to indicate how the results of [HOV 10] can be applied to an explicitly given system. The main questions we wish to address in this respect are the following.

1) Determine the stability domain in parameter space of the original system.

2) Locate the singularities on the boundary of the stability domain.

3) Identify Hamiltonian and reversible subsystems as many applications can be considered as perturbations of such systems.

8.2. Methods

8.2.1. *Centralizer unfolding*

Recall from section 8.1.2 that a map in $\mathrm{gl}(4,\mathbb{R})$ with a double pair of semi-simple eigenvalues $\pm i$ is equivalent to L as in equation [8.2]

$$L = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here, we are interested in an open neighborhood of L in $\mathrm{gl}(4,\mathbb{R})$. Note that such a neighborhood is 16-dimensional. By recognizing equivalent maps, we can greatly reduce the dimension. Each map in such a neighborhood is equivalent to an element of a transversal to the $\mathbf{Gl}(4,\mathbb{R})$-orbit of L. This is the idea of a universal unfolding of L and it has dimension 8 in this case. Among the universal unfoldings, the centralizer unfolding can be considered as a linear variety $L+\mathbf{m}$ in which $\mathbf{m}$ is a linear space, but at the same time a Lie algebra. By recognizing equivalence classes in the centralizer unfolding, we can further reduce the dimension to 5. Using homogeneity properties, we can even reduce the dimension to 3. To give the centralizer unfolding, we first present a convenient basis for $\mathrm{gl}(4,\mathbb{R})$.

LEMMA 8.1.– The matrices $\langle M_1,\dots,M_8,P_1,\dots,P_8\rangle$ form a basis of $\mathrm{gl}(4,\mathbb{R})$, where

$$\begin{aligned}
&M_1 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad M_2 = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad M_3 = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \\
&M_5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}, \quad M_8 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \\
&P_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P_2 = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}, \quad P_3 = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \\
&P_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix}, \quad P_7 = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}, \quad P_8 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}
\end{aligned} \qquad [8.4]$$

and the two-by-two matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

COMMENT 8.1.– Note that $M_1 = Id$ and $M_5 = L$. Furthermore, M_2, M_3 and M_4 are equivalent and have eigenvalues $\{1,-1,1,-1\}$. Similarly, M_5, M_6, M_7 and M_8 are equivalent, having imaginary eigenvalues $\{i,-i,i,-i\}$.

The first result is that a centralizer unfolding of L is given by the following.

PROPOSITION 8.1.– $\mathscr{L}(\mu) = L + \sum_{i=1}^{8} \mu_i M_i$, with parameters $\mu_1, \ldots, \mu_8 \in \mathbb{R}$ is a centralizer unfolding of L. The codimension of the unfolding is 8. The matrices M_i have the property $[M_i, L] = 0$. Thus, the Lie-algebra **m** mentioned in the beginning of this section is the linear subspace of $\mathrm{gl}(4,\mathbb{R})$ spanned by the matrices $\{M_1, \ldots, M_8\}$. Again we have a group of similarity transformations acting on **m** and this action turns out to be an $SO(3)$-action. All information about eigenvalues and (in)stability domains must be a function of the *invariants* of this action, or more precisely of the *generators* of the invariants.

LEMMA 8.2.– Let $x = (\mu_2, \mu_3, \mu_4)$, $y = (\mu_6, \mu_7, \mu_8)$ and $B = \mathrm{diag}(1,1,-1)$, then the *generators* of the *invariants* of the $SO(3)$-action on **m** are $\pi_1 = ||x||^2$, $\pi_2 = ||y||^2$, $\pi_3 = \langle x, By \rangle$, $\pi_4 = \mu_1$ and $\pi_5 = \mu_5$ with inequalities: $\pi_1 \geq 0$, $\pi_2 \geq 0$ and $\pi_3^2 \leq \pi_1 \pi_2$. Although the generators of the invariants are useful for actual computations given the centralizer unfolding, we cannot easily use them to analyze the boundary of the stability domain since the map $(x,y) \mapsto (\pi_1, \ldots, \pi_5)$ is not regular everywhere. To that end, we use a reduced centralizer unfolding.

PROPOSITION 8.2.– Every element of the centralizer unfolding $\mathscr{L}$ is equivalent to an element of a reduced centralizer unfolding. An example being $\mathscr{L}_r$ given by

$$\mathscr{L}_r(\nu) = L + \nu_1 M_1 + \nu_2 M_4 + \nu_3 M_6 + \nu_4 M_8 + \nu_5 M_5, \qquad [8.5]$$

with $\nu_i \in \mathbb{R}$. A reduced centralizer unfolding is not unique. The choice in the lemma will suffice to analyze the boundary of the stability domain in the next section. The proofs of lemma 8.1, propositions 8.1 and 8.2 can be found in [HOV 10]. The proof of lemma 8.2 will be the subject of a separate publication.

8.2.2. *Stability domain*

The stability domain is most easily characterized by its boundary. In our case, this amounts to the condition that a map $A \in \mathrm{gl}(4,\mathbb{R})$ is an element of the boundary if A has at least one pair of imaginary eigenvalues and the real parts of the other eigenvalues are non-positive. Every map in a small neighborhood of L is equivalent to an element of the centralizer unfolding of L. Therefore, we may safely restrict ourselves to the intersection of the stability domain and the centralizer unfolding of L. Then, we may characterize the stability domain and its boundary as follows.

THEOREM 8.1.– Let π be the generators of the invariants from lemma 8.2 and let F_g be the function $F_g(\pi) = \pi_4^4 + \pi_4^2(\pi_2 - \pi_1) - \pi_3^2$. Then, the stability domain near L is characterized by $F_g(\pi) > 0$ and $\pi_4 < 0$ and its boundary by $F_g(\pi) = 0$ and $\pi_4 \leq 0$.

This characterization will be most suited for computations with a given unfolding. To analyze the boundary of the stability domain, we make one further reduction. We may homogenize the unfolding in the following sense. Let us define $\mathscr{H}(\mu) = \sum_{i=1}^{8} \mu_i M_i$, then $\mathscr{L}(\mu) = \mathscr{H}(\mu_1,\mu_2,\mu_3,\mu_4, 1+\mu_5,\mu_6,\mu_7,\mu_8)$, because $L = M_5$. We call $\mathscr{H}$ the *homogeneous unfolding* of L. Similarly, we have a *homogeneous reduced unfolding* of L. Now stability of $t\mathscr{H}(\mu) = \mathscr{H}(t\mu)$ is equivalent to stability of $\mathscr{H}(\mu)$ for $t \neq 0$. Thus, we may restrict the parameters μ to a sphere. With the observation that $\mathscr{L}_r(\nu) = (1+\nu_5)L + \nu_1 M_1 + \nu_2 M_4 + \nu_3 M_6 + \nu_4 M_8$, we may restrict the parameters ν to the 3-sphere $\nu_1^2+\nu_2^2+\nu_3^2+\nu_4^2 = 1$. Then, we have an alternative theorem.

THEOREM 8.2.– Let F_r be the function $F_r(\nu) = (\nu_1^2-\nu_2^2)(\nu_1^2+\nu_4^2)+\nu_1^2\nu_3^2$. Then, the stability domain near L is characterized by $\{\mathscr{L}_r(\nu) \mid F_r(\nu) > 0, \nu_1 \leq 0, \nu_1^2+\nu_2^2+\nu_3^2+\nu_4^2 = 1\}$.

For the proofs of theorems 8.1 and 8.2, we again refer to [HOV 10]. To visualize the stability domain of L, we use the reduced centralizer unfolding. Let $\nu = (x,z)$ with $(x,z) \in \mathbb{R}^3 \times \mathbb{R}$ and $||x||^2 + z^2 = 1$. We use a blow-up of S^3 to D^3, namely $(x,z) \mapsto \arccos(z)\frac{x}{||x||}$. By this map, the point $(0,0,0,1)$ outside the stability domain is blown-up to a 2-sphere. The result is shown in Figure 8.2.

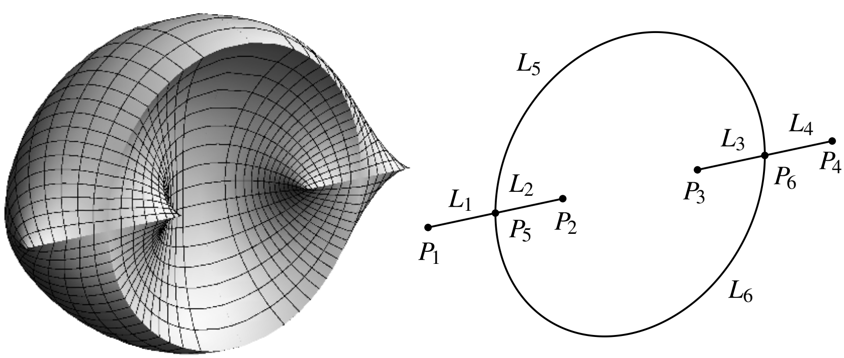


Figure 8.2. *Boundary of stability domain in S^3 blown-up to D^3. See text for an explanation of $P_1,\ldots,L_6$*

As mentioned in the beginning of this section, the matrix $A(\mu)$ has at least one pair of imaginary eigenvalues and the real parts of the other eigenvalues are non-positive. Such an *eigenvalue configuration* is denoted $\gamma_-\beta$. Almost all points of the boundary are of this type. Thus, in the full system, equation [8.1], we expect a Hopf bifurcation. In exceptional points, all eigenvalues are imaginary with configuration $\beta_1\beta_2$. This occurs on the lines $L_1,\ldots,L_6$. At these points, we expect

Hopf-Hopf bifurcations in the full system. These bifurcations are rather complicated even in the absence of resonance. However, in the linear analysis, they do not play a special role except at 1:1 resonance that we find at the points $P_1,\ldots,P_4$. Here, we have a nilpotent 1:1 resonance, semi-simple 1:1 resonance only occurs at $\nu = 0$. The boundary of the stability domain has a "Whitney umbrella" singularity at the points $P_1,\ldots,P_4$. This implies that under a generic perturbation, the system becomes unstable.

THEOREM 8.3.– Let $\dot{x} = F_\mu(x)$ be the differential equation as in equation [8.1]. We consider the stationary point $x = 0$. Suppose that μ has a value such that $A(\mu) = DF_\mu(0)$ is nilpotent at 1:1 resonance. Then, under a generic perturbation, the stationary point $x = 0$ becomes unstable.

The boundary of the stability domain has a *Whitney stratification* (see [HOV 10] for a proof). We show the strata with exceptional eigenvalue configurations in Table 8.1 together with their characterization using the reduced centralizer unfolding and the generators of the $SO(3)$-invariants.

Strata	evc	ν	π
$L_1,\ldots,L_4$	$\beta_1\beta_2$	$\nu_1 = \nu_4 = 0$	$\pi_3 = \pi_4 = 0$
L_5, L_6	$\beta_1\beta_2$	$\nu_1 = \nu_2 = 0$	$\pi_1 = \pi_4 = 0$, $\pi_2 \neq 0$
$P_1,\ldots,P_4$	β^2	$\nu_1 = \nu_4 = 0, \nu_2 = \pm\nu_3$	$\pi_3 = \pi_4 = 0$, $\pi_1 = \pi_2 \neq 0$
P_5, P_6	$\beta_1\beta_2$	$\nu_1 = \nu_2 = \nu_4 = 0$	$\pi_1 = \pi_4 = 0$, $\pi_2 \neq 0$

Table 8.1. *Eigenvalue configurations (evc) on the strata $L_1,\ldots,L_6$, $P_1,\ldots,P_6$ near L*

COMMENT 8.2.– The central singularity in the linear system has codimension 8, i.e. the centralizer unfolding needs eight parameters. Despite this fact, the reductions we have been able to perform suggest that we will find the destabilization phenomenon in generic three-parameter systems. Note that we cannot conclude this from a linear system where the central singularity is a nilpotent 1:1 resonance because it has codimension 4. Either way, we could find non-degeneracy conditions, but we will not do so here. In the examples below, we will check the conditions of theorems 8.1 or 8.2.

8.2.3. *Mapping into the centralizer unfolding*

Every deformation of every map A equivalent to L can be mapped into the centralizer unfolding of L. This result is stated in theorem 8.4 and the proof provides an algorithm to compute this map up to any desired order. It will be central in our examples in section 8.3.

THEOREM 8.4.– [STANDARD FORM] – Let $A(\mu)$ be a smooth p-parameter family in $\mathrm{gl}(4,\mathbb{R})$ where $A(0)$ is semi-simple with eigenvalue quadruplet $\{i,-i,i,-i\}$. Furthermore, let $\mathscr{L}$ be the centralizer unfolding of L. Then, there is a sequence of similarity transformations and a smooth map ϕ_k taking $A(\mu)$ into $\mathscr{L}(\phi_k(\mu)) + \mathscr{O}(||\mu||^{k+1})$.

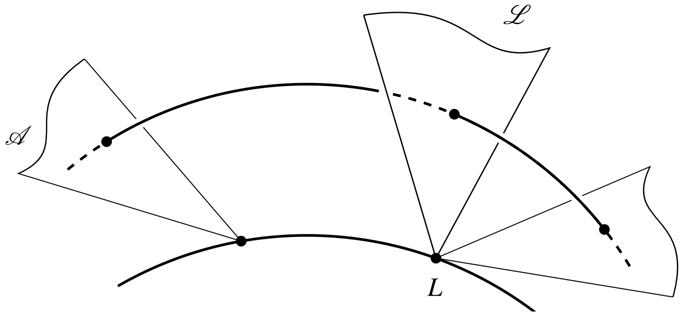


Figure 8.3. *Centralizer unfolding $\mathscr{L}$ of L and any deformation $\mathscr{A}$ of map equivalent to L*

The theorem follows from the following lemma and proposition.

LEMMA 8.3.– If $A(0)$ is semi-simple with eigenvalues $(i,-i,i,-i)$, then there exists a linear transformation T_0 such that $\mathscr{A}(\alpha) = T_0^{-1}A(\alpha)T_0$ is a deformation of L, that is $\mathscr{A}(0) = L$.

Since the proof of the lemma is straightforward, it will be omitted. The next proposition is in fact a normal form result.

PROPOSITION 8.3.– [NORMAL FORM] – Let $\mathscr{A}(\alpha) = L + X(\alpha)$ be a smooth deformation of L. Define $\mathscr{A}_0(\alpha) = L + X(\alpha)$ and inductively define $\mathscr{A}_k(\alpha) = T_k^{-1}(\alpha)\mathscr{A}_{k-1}(\alpha)T_k(\alpha)$, where $T_k(\alpha) = \mathrm{id} + U_k(\alpha)$ and U_k is homogeneous of degree k in α. Then

$$\mathscr{A}_k(\alpha) = L + \sum_{i=1}^{k} X_i + \mathscr{O}(||\alpha||^{k+1})$$

with X_i homogeneous of degree i in α and $[X_i(\alpha), L] = 0$ for $i = 1,\ldots,k$. In other words

$$\mathscr{A}_k(\alpha) = \mathscr{L}(\phi_k(\alpha)) + \mathscr{O}(||\alpha||^{k+1})$$

for some polynomial map ϕ_k.

The proof of the proposition is rather standard, but in the end we use some facts from linear unfoldings.

PROOF.– Since $\mathscr{A}(\alpha) = L + X(\alpha)$ is a smooth deformation, we may write $X(\alpha) = \sum_i X_i(\alpha)$, where X_i is homogeneous of degree i. Let $T_k(\alpha) = \mathrm{id} + U_k(\alpha)$, where U_k is homogeneous of degree k. Suppressing α dependence in the notation, we have

$$\begin{aligned} T_k^{-1}\mathscr{A}T_k &= (\mathrm{id}+U_k)^{-1}(L+X)(\mathrm{id}+U_k) \\ &= (\mathrm{id}-U_k)(L+X)(\mathrm{id}+U_k)+\mathscr{O}(k+1) \\ &= L+\sum_{i=1}^{k-1} X_i + X_k - [U_k, L] + \mathscr{O}(k+1). \end{aligned}$$

Because L is semi-simple, the linear operator $[\cdot, L]$ is also semi-simple and therefore we have the splitting $\mathrm{gl}(4,\mathbb{R}) = \ker([\cdot,L]) \oplus \mathrm{ran}([\cdot,L])$. Thus, we can find a U_k such that $X_k - [U_k, L] \in \ker([\cdot,L])$. This means that $X_i \in \ker([\cdot,L])$ for $i \in \{1,\cdots,k\}$. Now $\ker([\cdot,L]) = \langle M_1,\ldots,M_8\rangle$ and $\mathrm{ran}([\cdot,L]) = \langle P_1,\ldots,P_8\rangle$ (see equation [8.1]), therefore

$$\mathscr{A}_k(\alpha) = L + \sum_{i=1}^{8} \mu_i(\alpha) M_i + \sum_{i=1}^{8} \Big(\rho_{i,1}(\alpha) M_i + \rho_{i,2}(\alpha) P_i\Big),$$

where each μ is a polynomial of degree k and each ρ is a smooth function of order $\mathscr{O}(||\alpha||^{k+1})$. Thus, $\mathscr{A}_k(\alpha) = \mathscr{L}(\phi_k(\alpha)) + \mathscr{O}(||\alpha||^{k+1})$.

The results of the lemma and the proposition suffice to prove theorem 8.4.

PROOF.– Let T be the transformation $T = T_0 \circ T_1 \circ \cdots \circ T_k$ with $T_k, \ldots, T_1$ as in proposition 8.3 and T_0 as in lemma 8.3, then $T^{-1}A(\alpha)T$ takes the desired form.

COMMENT 8.3.– In practical situations, we may only need one step of the normal form procedure which amounts to a projection of the unfolding $\mathscr{A}(\alpha)$ on $\mathscr{L}(\mu)$, provided that $\mathscr{A}(0) = \mathscr{L}(0) = L$.

8.3. Examples

8.3.1. *Modulation instability*

A monochromatic plane wave with a finite amplitude propagating in a nonlinear and dispersive medium can be disrupted into a train of short pulses when the amplitude exceeds some threshold [BEN 67, BEN 67a, BES 66, ZAK 68]. This process develops due to an unbounded increase in the percentage modulation of the wave, i.e. instability of the carrier wave with respect to modulations [BRI 07, ZAK 09]. This is the fundamental *modulation instability*, known as the *Benjamin–Feir instability* in hydrodynamics [BEN 67] and the *Bespalov-Talanov instability* in nonlinear optics [BES 66].

Without dissipation, a slowly varying in time envelope C of the rapidly oscillating carrier wave is often described by the nonlinear Schrödinger equation (NLS)

$$iC_t + \alpha C_{\xi\xi} + \gamma|C|^2C = 0, \qquad [8.6]$$

where α and γ are positive real numbers and C is a complex amplitude depending on time t and position ξ in one space dimension [BRI 07, ZAK 68]. More realistic models of modulated waves take into account dissipative effects resulting in the modified NLS equation [BRI 07]

$$iC_t + (\alpha - ia)C_{\xi\xi} + ibC + (\gamma + ic)|C|^2C = 0 \qquad [8.7]$$

with a, b, and c being real. The term ibC is the viscosity of the medium, $iaC_{\xi\xi}$ is the rate of change of the group velocity of the carrier wave due to dissipation and $ic|C|^2C$ is the nonlinear damping [BRI 07].

8.3.1.1. *Linearization of the nonlinear Schrödinger equation*

Equation [8.6] has a solution in the form of a monochromatic wave

$$C = C_0 e^{ik\xi - i\omega t}, \qquad [8.8]$$

where the frequency of the modulation, ω, depends on the amplitude $C_0 = u_1 + iu_2$ and spatial wave number k as $\omega = \alpha k^2 - \gamma\|u\|^2$ with $u = (u_1, u_2)$.

To study the stability of the modulation, we linearize the dissipative NLS [8.7] about the basic traveling wave solution [8.8]. Assuming perturbations, periodic in x with wave number σ, we substitute their Fourier expansions into the linearized problem. Then, the σ-dependent modes decouple into four-dimensional subspaces for each harmonic, yielding a four-dimensional real linear differential equation, depending on several parameters [BRI 07]

$$\dot{x} = A(\nu)x \qquad [8.9]$$

with $x \in \mathbb{R}^4$, $\nu = (\alpha, \gamma, \sigma, k, u, a, b, c)$ and $A(\nu)$ a smooth parameter family in $\mathrm{gl}(4, \mathbb{R})$.

8.3.1.2. *The parameter family A*

We start with two properties of the family A. The original partial differential equation [8.6] is symplectic [ZAK 68, ZAK 09] for $a = b = c = 0$. The linearization is again symplectic, as it should be.

LEMMA 8.4.– The linearization [8.9] is symplectic when $a = b = c = 0$. The symplectic form ω is given by

$$\omega(x, y) = \langle x, \Omega y \rangle \text{ with } \Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}.$$

Note that $\Omega = M_8$, for M_8 as in equation [8.4]. Thus, we have, in particular, $(AM_8 - A^t M_8)_{a=b=c=0} = 0$, where A^t is the transpose of A with respect to the inner product $\langle \cdot, \cdot \rangle$.

The second property is about rotational symmetry of equation [8.9]. It is a crucial property because it allows us to use the unfoldings of the matrix L in section 8.2.1.

LEMMA 8.5.– Equation [8.9] is S^1-equivariant where the S^1-action is given by $g_t \cdot x = \mathrm{e}^{tL} x$, for $t \in \mathbb{R}$ and L as in equation [8.2].

PROOF.– In the complex equation [8.7] for C, we set $C = (u_1 + iu_2)$ with $u \in \mathbb{R}^2$. The linear part of equation [8.7], by its very nature, is equivariant under scalar multiplication, in particular under $C \mapsto iC$. The latter corresponds to $u \mapsto Ju$ where J is the 2×2 matrix defined in equation [8.4]. Thus, the linearization of the equation for u is J-equivariant. Taking a Fourier expansion of u and retaining only the lowest order modes, then the coefficients v and w of sin and cos, respectively, are elements of $\mathbb{R}^2$. Therefore, their respective equations are again J-equivariant. If now $x = (v_1, w_1, v_2, w_2) \in \mathbb{R}^4$, then the equation for x is L-equivariant. For further detail, we refer to [BRI 07, KIR 13a].

In section 8.2.1, we introduced the basis $\langle M_1, \dots, P_8 \rangle$ of $\mathrm{gl}(4, \mathbb{R})$, we use it here as well to decompose $A(\alpha, \gamma, \sigma, k, u, a, b, c)$. Thus, we obtain the following result.

LEMMA 8.6.– The parameter family A in equation [8.9] has a decomposition

$$A(\nu) = \mu_1 M_1 + \cdots + \mu_8 M_8 + \rho_1 P_1 + \cdots + \rho_8 P_8,$$

where μ_i and ρ_i depend on the parameters $\nu = (\alpha, \gamma, \sigma, k, u, a, b, c)$ and we find

$$\mu = (-b - a(\sigma^2 + k^2) - cu_1^2 - cu_2^2, -cu_1^2 - 2\gamma u_1 u_2 + cu_2^2, \gamma u_1^2 - 2cu_1 u_2 - \gamma u_2^2, -2ak\sigma,$$
$$- 2\alpha k\sigma, 0, 0, \alpha\sigma^2 - \gamma u_1^2 - \gamma u_2^2),$$
$$\rho = 0.$$

From property [8.9] it immediately follows that A must be a linear combination of the matrices $M_1, \ldots, M_8$ only. This implies that the parameter family A can be mapped into centralizer unfolding $\mathscr{L}$ of L by a transformation of parameters only. A closer inspection of the family A shows that it does not contain an element with double semi-simple imaginary eigenvalues. Only for $a = b = c = 0$ and $u_1 = u_2 = 0$, the limit $\sigma \to 0$ yields a matrix with a quadruplet of eigenvalues tending to a double pair of imaginary eigenvalues, but the imaginary parts also tend to zero with σ. However, in the unfolding $\mathscr{L}$, there is a subfamily of matrices with double nilpotent imaginary eigenvalues. As we will see, this case is relevant for the modulation instability. As a result, results can be derived from the stability analysis of $\mathscr{L}$.

8.3.1.3. *Asymptotic stability of the linearization in the presence of the a,b,c – dissipation*

The stability domain of the family A can almost immediately be obtained from the results for $\mathscr{L}$. However, here we prefer to use the so-called homogeneous unfolding $\mathscr{H}$. In the following, we take $\nu = (\alpha, \gamma, \sigma, k, u, a, b, c)$ and μ as in lemma 8.6 so that $A(\nu) = \mathscr{H}(\mu(\nu))$. We summarize the result in the next proposition.

PROPOSITION 8.4.– In the family A, $A(\nu)|_{a=b=c=0}$ is an element of the boundary of the stability domain. Furthermore, $A(\nu)|_{a=b=c=0}$ has a nilpotent 1:1 resonance if $\alpha\sigma^2 = 2\gamma||u||^2$. At these parameter values, the boundary of the stability domain has a "Whitney umbrella" singularity. Moreover, a generic small perturbation of $A(\nu)|_{a=b=c=0}$ at 1:1 resonance, even diffusive, will destabilize the stationary point of equation [8.9].

This is called the modulation instability of the wave equation [8.7] and its enhancement with dissipation [BRI 07]. In the following proof, we use the stability results from section 8.2.2.

PROOF.– To determine the stability of $A(\nu)$, we use the generators of the invariants as defined in lemma 8.2 and the stability criterion $F_g(\pi) > 0$ and $\pi_4 < 0$ from proposition 8.1. By the map $\nu \mapsto \mu(\nu)$, the generators become functions of ν. A straightforward computation yields $F_g(\pi)|_{a=b=c=0} = 0$ and $\pi_4|_{a=b=c=0} = 0$. Thus, using proposition 8.2, we conclude that $A(\nu)|_{a=b=c=0}$ is an element of the boundary of the stability domain. A double pair of imaginary eigenvalues occurs if $\pi_1 = \pi_2$, $\pi_3 = 0$ and $\pi_4 = 0$. Applying this to the family A, we find $(\pi_1 - \pi_2)|_{a=b=c=0} = \alpha\sigma^2(2\gamma||u||^2 - \alpha\sigma^2)$ and $\pi_3|_{a=b=c=0} = 0$. This double pair is semi-simple only if $\pi_1 = \pi_2 = 0$. From a similar computation for non-zero a, b and c, we find $F_g(\pi)|_{2\gamma||u||^2 - \alpha\sigma^2 = 0} = -\alpha^2 k^2 \sigma^6 a^2 + \mathscr{O}(4)$ and $\pi_4 = -(\sigma^2 + k^2)a - b - ||u||^2 c$, where $\mathscr{O}(n)$ contains terms of order equal to or higher than n in a, b and c. Since $F_g(\pi)|_{2\gamma||u||^2 - \alpha\sigma^2 = 0} < 0$, a generic small perturbation destabilizes the stationary point at 1:1 resonance.

8.3.1.4. *Imperfect merging of modes and threshold of the dissipative modulation instability*

Eigenvalues of the matrix A in the non-dissipative case when $a = b = c = 0$ are

$$\lambda = \pm i2\alpha k\sigma \pm i\sigma\sqrt{\alpha^2\sigma^2 - 2\alpha\gamma\|u\|^2}. \qquad [8.10]$$

At small amplitudes of the modulation, $\|u\| \neq 0$, the eigenvalues [8.10] are imaginary. With the increase in the amplitude, the modes with the opposite Krein (symplectic) sign collide at the threshold $\|u\| = \|u\|_i$, where [BES 66, ZAK 09]

$$\|u\|_i^2 = \frac{\alpha\sigma^2}{2\gamma}. \qquad [8.11]$$

At $\|u\| > \|u\|_i$ the double nilpotent imaginary eigenvalues split into complex-conjugate eigenvalues, one of which with a positive real part, that corresponds to the modulation instability in the ideal (undamped) case, see gray curves in figure 8.4(a). However, in the presence of dissipation, the growth rates exhibit an imperfect merging so that the upper branch becomes positive at a smaller value of $\|u\|$ than in the ideal situation (see black curves in Figure 8.4(a)). This is a manifestation of a *dissipation-induced instability* [BLO 94, BRI 07, KRE 07, KIR 10].

To uncover the details of the destabilization, we assume further in this section that $c = 0$. Then,

$$F_g = -4\sigma^2a^2k^2\gamma^2\|u\|^4 - 2\sigma^2\alpha\gamma r\|u\|^2 + r((b + a(k^2 + \sigma^2))^2 + \alpha^2\sigma^4), \qquad [8.12]$$

where $r = (b + a(k^2 + \sigma^2))^2 - 4a^2\sigma^2k^2$. Solving equation $F_g = 0$ with respect to b and retaining only linear in a terms we find

$$b = -a(k^2 + \sigma^2) \pm ak\frac{2\|u\|_i^2 - \|u\|^2}{\sqrt{\|u\|_i^2 - \|u\|^2}}\sqrt{\frac{2\gamma}{\alpha}}. \qquad [8.13]$$

For $a \ll b$, the linear approximation [8.13] to the stability boundary in the (a, b) plane yields a simple estimate of the instability threshold in the presence of dissipation

$$\|u\|_d \simeq \|u\|_i\left(1 - \frac{k^2\sigma^2}{2}\frac{a^2}{b^2}\right) \leq \|u\|_i. \qquad [8.14]$$

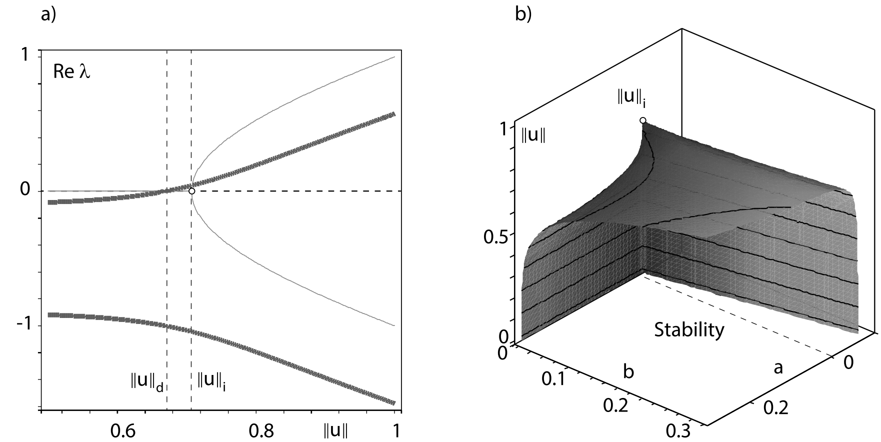


Figure 8.4. $\alpha = 1,\ \sigma = 1,\ \gamma = 1,\ k = 1,\ c = 0$: *a) Growth rates of the perturbation in the presence of dissipation (black) with* $a = 0.2$ *and* $b = 0.1$ *become positive at lower values of* $\|u\|$ *than in the undamped case (gray) with* $a = 0$ *and* $b = 0$ *(dissipation-enhanced modulation instability [BRI 07]). b) The threshold of the modulation instability* $(F_g = 0)$ *in the* $(a, b, \|u\|)$ *space with the "Whitney umbrella" singularity at* $a = 0$, $b = 0$ *and* $\|u\| = \|u\|_i = \sqrt{2}/2$

In the new variables $X = ak\sigma$, $Y = b\sqrt{2}$ and $Z = \|u\|_i - \|u\|_d$, equation [8.14] has the form $Z = X^2/Y^2$ that is canonical for the "Whitney umbrella" surface. Plotting the threshold $F_g = 0$ in the $(a, b, \|u\|)$ space, we see the singularity at $a = 0$, $b = 0$ and $\|u\| = \|u\|_i$ (Figure 8.4(b)). The presence of the singularity on the stability boundary explains why in the limit $b \to 0$ at every given $a \neq 0$, the resulting interval of modulation instability increases with respect to that at $a = b = 0$. This enhancement of the modulation instability with dissipation first observed in [BRI 07], is clearly seen in the approximation [8.14].

8.3.2. *Non-conservative gyroscopic system*

A rotating shaft is a classical model of rotor dynamics [SHI 68, CRA 70, CRA 95]. In [SHI 68] the shaft is modeled as rotating oscillator, i.e. as the mass m that is attached by two springs with the stiffness coefficients k_1 and k_2 and two dampers with the coefficients δ_1 and δ_2 to a coordinate system rotating at constant angular velocity Ω. A non-conservative positional force $r\beta$ acts on the mass. With u and v representing the displacements in the direction of the two rotating coordinate axes, respectively (then $r^2 = u^2 + v^2$), the system is governed by the two coupled equations [SHI 68]:

$$\begin{cases} m\ddot{u} + \delta_1\dot{u} - 2m\Omega\dot{v} + (k_1 - m\Omega^2)u + \beta v = 0, \\ m\ddot{v} + \delta_2\dot{v} + 2m\Omega\dot{u} + (k_2 - m\Omega^2)v - \beta u = 0. \end{cases} \qquad [8.15]$$

8.3.2.1. *The parameter family A*

To obtain equations [8.15] in the standard form $\dot{x} = A(\nu)x$, we first set $u_1 = u$, $v_1 = v$, $u_2 = \dot{u}$, $v_2 = \dot{v}$ and $x = (u_1, v_1, u_2, v_2)$. Furthermore, we redefine parameters so that m drops out:

$$\delta_1 = m\alpha_1,\ \delta_2 = m\alpha_2,\ k_1 = m(\omega^2 - \kappa),\ k_2 = m(\omega^2 + \kappa),\ \beta = m\gamma.$$

and we set $\nu = (\alpha_1, \alpha_2, \kappa, \gamma, \Omega)$ as (small) parameters, ω is a non-small constant. Finally, we apply a scaling transformation S to the differential equation with $S = \mathrm{diag}(\sqrt{\omega}, \sqrt{\omega}, \frac{1}{\sqrt{\omega}}, \frac{1}{\sqrt{\omega}})$. This yields

$$A(\nu) = \begin{pmatrix} 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \\ -\omega & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\Omega^2+\kappa}{\omega} & -\frac{\gamma}{\omega} & -\alpha_1 & 2\Omega \\ \frac{\gamma}{\omega} & \frac{\Omega^2-\kappa}{\omega} & -2\Omega & -\alpha_2 \end{pmatrix}$$

and $A(0) = \omega L$, with L as in equation [8.2]. The parameter family $A(\nu)$ has a unique decomposition in $\mathrm{gl}(4, \mathbb{R})$.

LEMMA 8.7.– In the decomposition $A(\nu) = \omega L + \mu_1 M_1 + \cdots + \mu_8 M_8 + \rho_1 P_1 + \cdots + \rho_8 P_8$, the μ and ρ are

$$\begin{aligned} \mu &= (-\frac{\alpha_1 + \alpha_2}{4}, \frac{\alpha_2 - \alpha_1}{4}, 0, \frac{\gamma}{2\omega}, -\frac{\Omega^2}{2\omega}, -\frac{\kappa}{2\omega}, 0, \Omega) \\ \rho &= -\mu \end{aligned}$$

8.3.2.2. *Stability of the family* $A(\nu)$

To apply the stability results of section 8.2.2, we first have to map the family $A(\nu)$ into the centralizer unfolding (see section 8.2.3). It will turn out that a first approximation is enough to draw conclusions about stability. Thus, from now on, we consider the equation

$$\dot{x} = \overline{A}(\nu) = \omega L + \mu_1 M_1 + \cdots + \mu_8 M_8. \qquad [8.16]$$

We summarize the result in the next proposition.

PROPOSITION 8.5.– The family $\overline{A}(\nu)$ has a nilpotent 1:1 resonance for the parameter values $(\alpha_1, \alpha_2, \kappa) = \pm(2\Omega, -2\Omega, \gamma)$. A generic small perturbation of these values destabilizes the stationary point of the system.

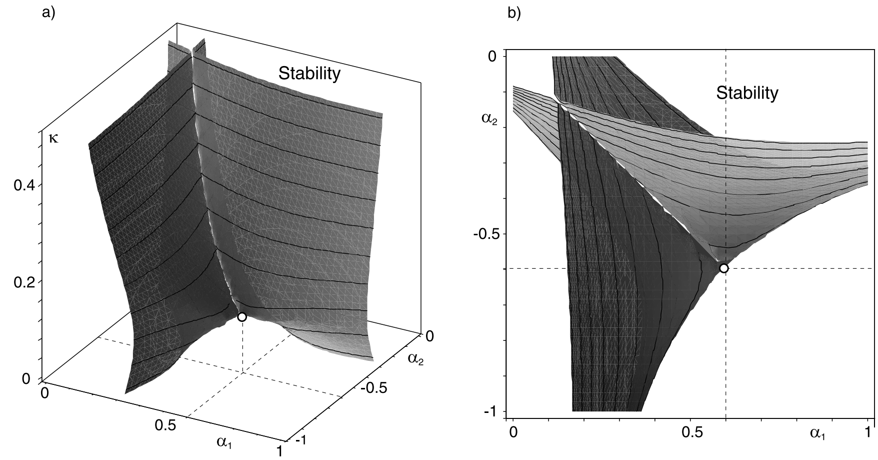


Figure 8.5. $\omega = 1$, $\Omega = 0.3$, $\gamma = 0.1$. *a) The surface* $F_g = 0$ *in the* $(\alpha_1, \alpha_2, \kappa)$ *space with the "Whitney umbrella" singularity (open circle) at* $\alpha_1 = 0.6$, $\alpha_2 = -0.6$ *and* $\kappa = 0.1$ *that bounds the domain of asymptotic stability of the rotating shaft. b) Its view from above*

PROOF.– The family $\overline{A}(\nu)$ has a nilpotent 1:1 resonance if $\pi_1 = \pi_2 \neq 0$ and $\pi_3 = \pi_4 = 0$. These conditions are fulfilled when $\nu_0 = (\alpha_1, \alpha_2, \kappa, \gamma, \Omega)$ with $(\alpha_1, \alpha_2, \kappa) = \pm(2\Omega, -2\Omega, \gamma)$. Now, we apply a small perturbation: we take $\nu_1 = \pm(2\Omega, -2\Omega, \gamma, 0, 0) + (\delta_1, -\delta_2, \varepsilon, 0, 0)$. Then, we check the stability condition from theorem 8.1 and we find $F_g(\pi(\nu_1)) = -\frac{1}{64}(\delta_1\gamma - \delta_2\gamma + 4\varepsilon\Omega)^2 + \mathscr{O}(3)$, where $\mathscr{O}(n)$ contains terms of order equal to or higher than n in δ_1, δ_2 and ε. Thus, for a generic small perturbation $F_g(\pi(\nu_1)) < 0$, so the stationary point at zero of the system becomes unstable.

Stability boundary $F_g = 0$ in the $(\alpha_1, \alpha_2, \kappa)$ space is shown in Figure 8.5. Non-conservative forces with $\gamma \neq 0$ produce an instability window around the origin that is distorted by the gyroscopic forces Figure 8.5(a)). As a result, the critical surface has self-intersections along the hyperbolic curves $\kappa\alpha_1 - 2\Omega\gamma = 0$ that lie in the plane $\alpha_1 + \alpha_2 = 0$. It is remarkable that on the curves all eigenvalues of the non-conservative gyroscopic system with dissipative and circulatory forces are purely imaginary. Almost everywhere on the curves of self-intersection, the eigenvalues are simple except for the points with the coordinates in the $(\alpha_1, \alpha_2, \kappa)$-space

$$(2\Omega, -2\Omega, \gamma), \quad (-2\Omega, 2\Omega, -\gamma), \qquad [8.17]$$

where the pure imaginary eigenvalues are double and have a Jordan block.

At $\kappa = 0$, the stability domain shown in Figure 8.5(a) does not contain the origin, in accordance with the Bottema–Lakhadanov–Karapetyan theorem that states that a gyroscopic system with non-conservative positional forces is generically unstable without dissipation [BOT 55, LAK 75]. However, our results illustrate Merkin's theorem on the destabilization of a potential system with the equal frequencies $(\kappa = 0)$ by non-conservative positional forces only $(\gamma \neq 0)$ when $\Omega = 0$ and $\alpha_1 = \alpha_2 = 0$ [BUL 11, KRE 06].

8.4. Conclusions

For a parametric family of four-dimensional linear dynamical systems determined by a matrix $A(\mu)$ with $A(0)$ in 1:1 semi-simple resonance, we have established that the central singularity on the stability boundary has codimension 8, i.e. the centralizer unfolding of the family needs eight parameters. By recognizing equivalence classes in the centralizer unfolding, we reduced the codimension to 5 and finally, by using the homogeneity properties, to 3. This allowed us to explicitly find the boundary of the stability domain and list all its singularities including six self-intersections and four "Whitney umbrellas". We have proposed an algorithm of approximation of the stability boundary near singularities and applied the results to the study of enhancement of the modulation instability with dissipation as well as to the study of stability of a non-conservative system of rotor dynamics.

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