Chapter 7

Representation of Wave Energy of a Rotating Flow in Terms of the Dispersion Relation

7.1. Introduction

In nature, there is a limited class of steady flow fields that are stable to three-dimensional (3D) disturbances. Being stable requires spatial symmetries, and amplification of waves by symmetry-breaking perturbations is a ubiquitous phenomenon. The 3D instability of a strained vortex tube, called the *Moore–Saffman–Tsai–Widnall (MSTW) instability* [MOO 75, TSA 76, ELO 01, FUK 03], belongs to this category. A family of neutrally stable oscillations of a circular cylindrical vortex tube, known as the inertial waves, is referred to as the Kelvin waves. The MSTW instability is a parametric resonance between two Kelvin waves whose azimuthal wave numbers are separated by two, which is caused by a perturbation breaking the circular symmetry of the streamlines into ellipses.

Krein's theory of Hamiltonian spectra accounts for this instability. In the language of eigenvalues, the spectral instability corresponds to the emergence of eigenvalues with positive real part through a collision of eigenvalues on the imaginary axis of the complex plane. A necessary condition for instability is that the two waves associated with the degenerate eigenvalue have energies of opposite sign [ARN 66a, ARN 66b, MOR 98]. Being fed by the symmetry-breaking perturbation, the positive- and negative-energy waves together increase in their amplitudes, without a change in the total energy. Thus, we are motivated to calculate the energy of Kelvin waves. In the presence of a basic flow, the traditional approach based on the Euler equations encounters a difficulty, because calculation of the energy of

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Kelvin waves of second order in wave amplitude requires the disturbance field to second order as well for evaluating the term coupling the basic and disturbance field, as is explained subsequently.

Given a steady flow $\boldsymbol{U}(\boldsymbol{x})$, we superimpose disturbance field $\boldsymbol{u}_\alpha$ of $O(\alpha)$ on it. The total velocity field is then

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}_\alpha. \quad [7.1]$$

The energy of a wave is defined as the excess energy relative to that of the basic flow when the wave is excited. By expanding $\boldsymbol{u}_\alpha$ in powers of α, the amplitude parameter, to second order, as

$$\boldsymbol{u}_\alpha = \alpha \boldsymbol{u}_1 + \frac{1}{2}\alpha^2 \boldsymbol{u}_2 + \cdots, \quad [7.2]$$

the wave energy E is expressed by

$$\begin{aligned} E &= \frac{1}{2}\int \boldsymbol{u}^2 \mathrm{d}V - \frac{1}{2}\int \boldsymbol{U}^2 \mathrm{d}V \\ &= \alpha \int \boldsymbol{U}\cdot\boldsymbol{u}_1 \mathrm{d}V + \frac{1}{2}\alpha^2 \int \left(\boldsymbol{u}_1^2 + \boldsymbol{U}\cdot\boldsymbol{u}_2\right)\mathrm{d}V. \end{aligned} \quad [7.3]$$

The term $O(\alpha)$ vanishes by taking some average, say average in time, and therefore the wave energy is $O(\alpha^2)$. It can be seen from the $O(\alpha^2)$-terms in [7.3] that the presence of the basic flow $\boldsymbol{U}$ requires a knowledge of disturbance field $\boldsymbol{u}_2$ to $O(\alpha^2)$. Gaining the second-order disturbance field $\boldsymbol{u}_2$ by integrating nonlinear partial differential equations would not be straightforward, because this problem belongs to singular perturbations.

A bypassing route is available from an analogy with the electromagnetism for which the electromagnetic energy is calculated from a derivative of the dielectric permittivity and the magnetic permeability [LAN 84]. For the Vlasov–Poisson system, a theoretical foundation was provided from the viewpoint of the Hamiltonian system to the relation of the plasma energy with the *dielectric energy* [MOR 92]. Cairns [CAI 79] convincingly showed that for irrotational waves on a free surface of an inviscid incompressible fluid the wave energy is provided by a derivative of a certain form of the dispersion relation. However, for vortical flow, the entire vortical region plays the role of the interface, and a procedure parallel to Cairns' cannot be used. Fukumoto [FUK 03] manufactured, by *trial and error*, a plausible formula, in the form of a derivative of the dispersion relation, for the energy of a wave of normal form excited on the core of the Rankine vortex in an infinite space. The Rankine vortex is a circular cylindrical vortex of uniform vorticity and we introduce cylindrical coordinates (r, θ, z) with the z-axis lying on the axis of the cylinder. For a Kelvin wave, a normal mode proportional to $\exp[i(m\theta + k_0 z)]$ with m $(\in \mathbb{Z})$ and k_0 $(\in \mathbb{R})$ being the azimuthal and the axial wave numbers, respectively, the dispersion relation is written as $D(\omega_0; m, k_0) = 0$. The energy of the Kelvin wave of second order in wave amplitude α is

$$E = -\frac{\pi}{2}\omega_0 \frac{\partial D}{\partial \omega_0}\alpha^2. \qquad [7.4]$$

As far as the sign of energy is concerned, the resulting formula seems to be consistent with the MSTW instability, but its relation to the wave energy, including the magnitude, is yet to be verified. For two-dimensional shear flows, a proof is given to the relation that holds between the energy and a derivative of the dispersion relation not only for discrete spectra but also for continuous spectra, by judiciously defining the latter [HIR 08a, HIR 08b].

A steady incompressible Euler flow is characterized as a state of the maximum of the total kinetic energy with respect to disturbances constrained to an isovortical sheet. By exploiting this criticality, we are able to express the energy of waves to second order in amplitude only in terms of first-order disturbances. Use of the Lagrangian variables is advantageous for representing the isovortical disturbances. By this means, we obtained the formula for the energy of waves in a general context [FUK 08, FUK 10], and its various variants [FUK 11]. A major purpose of this chapter is to give an outline of the link of the wave energy, in the context of Kelvin waves confined in a circular cylinder, with a derivative of the dispersion relation.

The benefit of using the Lagrangian variables is to obtain the mean flow, of second order in amplitude, induced by the nonlinear interaction of Kelvin waves, again only in terms of first-order disturbances [FUK 08]. The existence of a mean flow in the azimuthal direction is expected, as this component is requisite for a negative-energy mode. Other than this component, an axial mean flow of second-order was found, which had been overlooked in the previous studies. The axial mean flow is not induced in two dimensions, and is peculiar to three dimensions. Having the uniquely determined mean flow of second order, we are able to unambiguously determine all the coefficients of the weakly nonlinear amplitude equations of third order [MIE 10, FUK 10, FUK 13].

We begin in section 7.2 with a concise description of derivation of the wave energy using the Lagrangian displacement field. Kelvin waves are recalled in section 7.3. Then in section 7.4, we establish the relation of the formula of the wave energy with a derivative of the dispersion relation with respect to the frequency. In section 7.5, we discuss the derivation of the mean flow induced by the nonlinear interactions of Kelvin waves and their utility for deriving the amplitude equations to third order.

7.2. Lagrangian approach to wave energy

A systematic derivation of wave energy is based on Kelvin–Arnold's theorem [ARN 66b] that a steady state of the Euler flows of an incompressible fluid is an extremal of the kinetic energy with respect to isovortical disturbances. The isovortical disturbances are easily constructed in the framework of the Lagrangian description, with which formulas of the wave energy were manipulated [FUK 08, FUK 10]. A brief outline of their derivation is given.

We assume that the fluid is incompressible as well as inviscid, and consider the density of fluid to be unity. The motion of an inviscid incompressible fluid is regarded as an orbit on SDiff($\mathscr{D}$), the group of volume-preserving diffeomorphisms of the domain $\mathscr{D} \subset \mathbb{R}^3$ [ARN 66a]. Its Lie algebra, $\mathfrak{g}$ is the velocity field of the fluid. A one-parameter subgroup of $\varphi_t \in$ SDiff($\mathscr{D}$) and its generator $u(t) \in \mathfrak{g}$ are linked by the definition

$$u(t_0) = \frac{\partial}{\partial t}\bigg|_{t_0} \left(\varphi_t \circ \varphi_{t_0}^{-1}\right). \qquad [7.5]$$

Suppose that the orbit φ_t of the basic flow is disturbed, at each instant t, to $\varphi_{\alpha,t} \circ \varphi_t$ by a near-identity map $\varphi_{\alpha,t}$ labeled with a small parameter $\alpha (\in \mathbb{R})$. A generator $\xi_\alpha(t) \in \mathfrak{g}$ exists for it, defined by $\varphi_{\alpha,t} = \exp \xi_\alpha(t)$.

The disturbance velocity field $u_\alpha(t)$ is calculated from

$$u_\alpha(t_0) = \frac{\partial}{\partial t}\bigg|_{t_0} \left(\varphi_{\alpha,t} \circ \varphi_t \circ \varphi_{t_0}^{-1} \circ \varphi_{\alpha,t_0}^{-1}\right). \qquad [7.6]$$

Use of a geometric setting, combined with symbolic manipulation of the Lie algebra, facilitates perturbation expansions, in powers of α, of the Lagrangian field to a higher order [HIR 09, FUK 10, HIR 11].

Translation into the language of the vector calculus is straightforward. Given a steady basic flow $\boldsymbol{U}_0(\boldsymbol{x})$, an orbit $\boldsymbol{x}(t)$ of a fluid particle constituting this basic flow, a realization of φ_t, is defined by $\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t = \boldsymbol{U}_0(\boldsymbol{x}(t))$. Suppose that the particle position $\boldsymbol{x}$ is disturbed to $\varphi_{\alpha,t}(\boldsymbol{x}) = \exp\big(\boldsymbol{\xi}_\alpha(\boldsymbol{x},t)\big)\boldsymbol{x}$. We expand the Lagrangian displacement field $\boldsymbol{\xi}_\alpha$ and correspondingly $\boldsymbol{u}_\alpha(\boldsymbol{x},t)$ in a power series in α as

$$\boldsymbol{\xi}_\alpha(\boldsymbol{x},t) = \alpha\boldsymbol{\xi}_1(\boldsymbol{x},t) + \frac{\alpha^2}{2}\boldsymbol{\xi}_2(\boldsymbol{x},t) + \cdots, \quad [7.7]$$

$$\boldsymbol{u}_\alpha(\boldsymbol{x},t) = \alpha\boldsymbol{u}_1(\boldsymbol{x},t) + \frac{\alpha^2}{2}\boldsymbol{u}_2(\boldsymbol{x},t) + \cdots. \quad [7.8]$$

The series development of [7.6] becomes, to $O(\alpha^2)$,

$$\frac{\partial\boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{U}_0 = \boldsymbol{u}_1, \quad [7.9]$$

$$\frac{\partial\boldsymbol{\xi}_2}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\boldsymbol{\xi}_2 - (\boldsymbol{\xi}_2\cdot\nabla)\boldsymbol{U}_0 + (\boldsymbol{u}_1\cdot\nabla)\boldsymbol{\xi}_1 - (\boldsymbol{\xi}_1\cdot\nabla)\boldsymbol{u}_1 = \boldsymbol{u}_2. \quad [7.10]$$

The second-order equation [7.10] was derived in our recent papers [FUK 08, FUK 10]. It is worth noting that the geometric approach is crucial for deriving [7.10] and higher order equations.

The requirement that the disturbance be *isovortical* or *kinematically accessible* is determined by preservation of the vorticity flux across an arbitrary infinitesimal material surface as represented, in the local Cartesian coordinates, by $\omega_x \mathrm{d}y\wedge\mathrm{d}z + \omega_y \mathrm{d}z\wedge\mathrm{d}x + \omega_z \mathrm{d}x\wedge\mathrm{d}y$ or, equivalently, by preservation of the circulation with respect to an arbitrary material loop contained in the domain [ARN 66a, ARN 66b, HOL 09]. This requirement brings in a series representation of the disturbance vorticity field $\boldsymbol{\omega}_\alpha$ [FUK 08]. Define the vector potential $\boldsymbol{v}_\alpha$ for $\boldsymbol{\omega}_\alpha$ by $\boldsymbol{\omega}_\alpha = \nabla\times\boldsymbol{v}_\alpha$, allowing for the freedom of the gauge transformation. The vector field $\boldsymbol{v}_\alpha$ belongs to $\mathfrak{g}^*$, dual of $\mathfrak{g}$. If $\boldsymbol{v}_\alpha(\boldsymbol{x},t)$ is expanded in a power series in α as

$$\boldsymbol{v}_\alpha(\boldsymbol{x},t) = \alpha\boldsymbol{v}_1(\boldsymbol{x},t) + \frac{\alpha^2}{2}\boldsymbol{v}_2(\boldsymbol{x},t) + \cdots, \quad [7.11]$$

the isovortical disturbance is represented, order by order, with the use of the Lagrangian displacement field, as

$$\boldsymbol{v}_1 = \mathscr{P}[\boldsymbol{\xi}_1\times\boldsymbol{\omega}_0], \quad [7.12]$$

$$\boldsymbol{v}_2 = \mathscr{P}\left[\boldsymbol{\xi}_1\times\left(\nabla\times(\boldsymbol{\xi}_1\times\boldsymbol{\omega}_0)\right) + \boldsymbol{\xi}_2\times\boldsymbol{\omega}_0\right], \quad [7.13]$$

where $\omega_0 = \nabla\times\boldsymbol{U}_0$ is the vorticity of the basic field and $\mathscr{P}$ is an operator projecting to the solenoidal vector field complying with the boundary condition.

The kinematics of vorticity disregards the concepts of the energy, the momentum and the force. The above formulas, along with the mass conservation, constitute the kinematics of the vorticity. The conservation laws of the energy and the momentum, in a manner compatible with the isovortical property, are retrieved by the identification [HOL 98]

$$\boldsymbol{v}_\alpha(\boldsymbol{x},t) = \boldsymbol{u}_\alpha(\boldsymbol{x},t). \tag{7.14}$$

With this identification in [7.9] and [7.10], the time evolution of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ is made to be compatible with the Euler equations. For later use, we write down the evolution equation of $\boldsymbol{\xi}_1$

$$\frac{\partial \boldsymbol{\xi}_1}{\partial t} = \mathscr{P}[\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_0] + \nabla \times (\boldsymbol{U}_0 \times \boldsymbol{\xi}_1). \tag{7.15}$$

We now calculate the wave energy E to second order in α as $E = \alpha E_1 + \alpha^2 E_2/2 + \cdots$. At first order,

$$E_1 = \int \boldsymbol{v}_1 \cdot \boldsymbol{U}_0 \mathrm{d}V = \int \boldsymbol{\xi}_1 \cdot (\boldsymbol{\omega}_0 \times \boldsymbol{U}_0)\, \mathrm{d}V = -\int \boldsymbol{\xi}_1 \cdot \frac{\partial \boldsymbol{U}_0}{\partial t} \mathrm{d}V = 0, \tag{7.16}$$

because $\boldsymbol{U}_0$ is a steady vector field. It is noteworthy that E_1 exactly vanishes without taking any average, a manifestation of Kelvin–Arnold's theorem. In a similar way, the second-order energy E_2 is manipulated as

$$\begin{aligned} E_2 &= \int (\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{U}_0)\, \mathrm{d}V = \int \Big\{ \boldsymbol{\xi}_2 \cdot (\boldsymbol{\omega}_0 \times \boldsymbol{U}_0) \\ &\quad + (\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_0) \cdot [\mathscr{P}[\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_0] + \nabla \times (\boldsymbol{U}_0 \times \boldsymbol{\xi}_1)] \Big\} \mathrm{d}V \\ &= \int \boldsymbol{\omega}_0 \cdot \left(\frac{\partial \boldsymbol{\xi}_1}{\partial t} \times \boldsymbol{\xi}_1 \right) \mathrm{d}V, \end{aligned} \tag{7.17}$$

where the steadiness of $\boldsymbol{U}_0$ and [7.15] have been used. In the same way as at $O(\alpha)$, the wave energy $\alpha^2 E_2/2$ rules out the disturbance field $\alpha^2 \boldsymbol{\xi}_2/2$ for a steady basic flow. This formula has been known over many years [ARN 66b, WU 06]. For a time-periodic disturbance $\boldsymbol{\xi}_1(t) = \boldsymbol{\xi}_1(t+T)$, [7.17] is converted into a convenient form [FUK 11]

$$E_2 = \frac{1}{T} \int_0^T E_2 \mathrm{d}t = 2 \int \frac{\partial \boldsymbol{\xi}_1}{\partial t} \cdot \left[\frac{\partial \boldsymbol{\xi}_1}{\partial t} + (\boldsymbol{U}_0 \cdot \nabla) \boldsymbol{\xi}_1 \right] \mathrm{d}V. \tag{7.18}$$

After describing waves on a flow of rigid-body rotation, in the following section, we pursue the link of [7.17] to a derivative of the dispersion relation.

7.3. Kelvin waves

We briefly recall the Kelvin waves, a family of neutrally stable linear oscillations, in a confined geometry [MIE 10]. We take, as a basic flow, the rigid-body rotation of an inviscid incompressible fluid confined in a cylinder of circular cross-section and of unit radius.

Let the r and the θ components of the two-dimensional basic velocity field $\boldsymbol{U}_0$ be U_0 and V_0, respectively, and the pressure be P_0. The basic flow is confined in $r \leq 1$, with the velocity and the pressure field given by

$$U_0 = 0, \ \ V_0 = r, \ \ P_0 = r^2/2 - 1. \qquad [7.19]$$

The vorticity of the basic flow is given by $\omega_0 = \nabla \times \boldsymbol{U}_0 = (0,0,2)$.

Take, as the disturbance velocity field to $O(\alpha)$, $\boldsymbol{u}_\alpha = \alpha \boldsymbol{u}_1 + \cdots$, a normal mode with azimuthal wave number $m (\in \mathbb{Z})$ and axial wave number k_0,

$$\boldsymbol{u}_1 = A_m \boldsymbol{u}_1^{(m)}(r) \mathrm{e}^{i(m\theta + k_0 z - \omega_0 t)} + c.c., \qquad [7.20]$$

where the amplitude A_m is a complex number and ω_0 is the frequency. In order for the velocity to be real, the complex conjugate term as designated by c.c. should be supplemented. The disturbance field $\boldsymbol{u}_1$ is governed by the linearized Euler equations, and, for the normal mode [7.20], they are expressed, in tidy form, in terms of the Bessel functions. For the basic flow [7.19], the displacement field $\boldsymbol{\xi}_1$ is found by integrating [7.9] simply to be

$$\boldsymbol{\xi}_1 = \frac{iA_m}{\omega_0 - m} \boldsymbol{u}_1^{(m)}(r) \mathrm{e}^{i(m\theta + k_0 z - \omega_0 t)} + c.c. \qquad [7.21]$$

A direct approach for seeking the link of the wave energy to the dispersion relation is to start from [7.15]. For the normal mode of the form $\boldsymbol{\xi}_1(r,\theta,z,t) = \hat{\boldsymbol{\xi}}(r;\omega_0,m,k_0)\mathrm{e}^{i(m\theta + k_0 z - \omega_0 t)} + c.c.$, [7.15] collapses to a single equation for the r-component $\hat{\xi}_r$ of $\boldsymbol{\xi}_1$ as

$$\mathscr{E}(\omega_0)r\hat{\xi}_r = -\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{r(\omega_0-m)^2}{m^2+k^2r^2}\frac{d}{dr}(r\hat{\xi}_r)\right]+\left[\frac{(\omega_0-m)^2}{r}\right.$$
$$\left.+\frac{4k^2rm(\omega_0-m)}{(m^2+k^2r^2)^2}-\frac{4k^2r}{m^2+k^2r^2}\right]r\hat{\xi}_r=0, \quad [7.22]$$

subject to the boundary conditions

$$\lim_{r\to 0}[r\hat{\xi}_r(r;\omega_0,m,k_0)] = \hat{\xi}_r(1;\omega_0,m,k_0) = 0. \quad [7.23]$$

Equation [7.22], abbreviated as $\mathscr{E}(\omega_0)r\hat{\xi}_r = 0$, is a special case of the equation written out in the context of the magnetohydrodynamics (MHD) wave in [GOE 10] and [KHA 08]. Once $\hat{\xi}_r$ is solved, the remaining components of $\hat{\boldsymbol{\xi}}$ are derived by

$$\hat{\xi}_\theta = i\frac{1}{m^2+k^2r^2}\left[m\frac{d}{dr}(r\hat{\xi}_r)-\frac{2k^2r}{\omega_0-m}r\hat{\xi}_r\right], \quad [7.24]$$

$$\hat{\xi}_z = i\frac{1}{m^2+k^2r^2}\left[kr\frac{d}{dr}(r\hat{\xi}_r)+\frac{2km}{\omega_0-m}r\hat{\xi}_r\right]. \quad [7.25]$$

The general solution of [7.22] complying with the first of [7.23] is easily obtained as

$$\hat{\xi}_r = \frac{(\omega_0-m)}{\sqrt{4-(\omega_0-m)^2}}\left[-(\omega_0-m)\eta_0 J_{m+1}(\eta_0 r)\right.$$
$$\left.+\frac{m}{r}(\omega_0-m-2)J_m(\eta_0 r)\right], \quad [7.26]$$

where the radial wave number is defined as $\eta_0 = \eta(\omega_0;m,k_0)$ by

$$\eta(\Omega;m,k_0) = k_0\sqrt{\frac{4}{(\Omega-m)^2}-1} \quad (\Omega\in\mathbb{C}). \quad [7.27]$$

Enforcement of the boundary condition $\hat{\xi}_r(1) = 0$ on [7.26] yields the dispersion relation for (ω_0,m,k_0)

$$-(\omega_0-m)\eta_0 J_{m+1}(\eta_0)+m(\omega_0-m-2)J_m(\eta_0) = 0. \quad [7.28]$$

When [7.17] is substituted by the eigenfunction $\boldsymbol{\xi}_1$ of a Kelvin wave, the wave energy E_2 per unit length in z is written as

$$E_2 = \omega_0 \mu_0, \tag{7.29}$$

where, with the use of complex conjugate $\overline{\hat{\xi}_r(r)} = \hat{\xi}_r(r)$ and $\overline{\hat{\xi}_\theta(r)} = -\hat{\xi}_\theta(r)$,

$$\mu_0 = 2\pi \int_0^1 2i\hat{\xi}_r\hat{\xi}_\theta r\mathrm{d}r. \tag{7.30}$$

Note that μ_0 is nothing but the action variable of this eigenmode. The first factor ω_0 in [7.29] originates from the time derivative in [7.17].

Since the eigenmode is periodic in time with period $T = 2\pi/\omega_0$, [7.18] is called for, yielding

$$\mu_0 = 2\pi \int_0^1 \left[(\omega_0 - m)|\hat{\boldsymbol{\xi}}|^2 - 2i\hat{\xi}_r\hat{\xi}_\theta\right] r\mathrm{d}r. \tag{7.31}$$

Comparison of [7.30] with [7.31] shows

$$\mu_0 = 2\pi \frac{\omega_0 - m}{2} \int_0^1 |\hat{\boldsymbol{\xi}}|^2 r\mathrm{d}r. \tag{7.32}$$

This representation manifestly shows that the sign of μ_0 coincides with that of $\omega_0 - m$; the wave action is positive (respectively, negative) for a cograde (respectively, a retrograde) mode for which the rotational frequency ω_0/m of the wave is higher (respectively, lower) than that of the basic flow $V_0/r = 1$. The sign of the wave energy $E_2 = \omega_0\mu_0$ depends, in addition, on whether $\omega_0 > 0$ or $\omega_0 < 0$.

Substituting from [7.24] and [7.25] and performing partial integration, [7.30] or [7.32] is reduced, with the aid of [7.22] and [7.23], to an integral of $\hat{\xi}_r$ only

$$\mu_0 = 2\pi \int_0^1 \left[\frac{4k^2 r}{(\omega_0 - m)(m^2 + k^2 r^2)} - \frac{2mk^2 r}{(m^2 + k^2 r^2)^2}\right] (r\hat{\xi}_r)^2 \mathrm{d}r. \tag{7.33}$$

7.4. Wave energy in terms of the dispersion relation

The wave energy and, more directly, the wave action are subject to be linked with a derivative of the dispersion relation [LAN 84, MOR 92, CAI 79]. We will show that this is also true for Kelvin waves. We introduce the Laplace transform (one-sided Fourier transform) $\Xi(r,\Omega;m,k_0)$ of $r\hat{\xi}_r(r;\omega_0,m,k_0)e^{-i\omega_0 t}$ by

$$\Xi(r,\Omega) = \int_0^\infty [r\hat{\xi}_r(r;\omega_0)e^{-i\omega_0 t}]e^{i\Omega t}\mathrm{d}t = \frac{i}{\Omega-\omega_0}r\hat{\xi}_r(r;\omega_0), \qquad [7.34]$$

where $\Omega \in \mathbb{C}$ and the assumption is made of $\mathrm{Im}[\Omega] > 0$. Its inverse transform is given by

$$r\hat{\xi}_r(r;\omega_0)e^{-i\omega_0 t} = -\frac{1}{2\pi}\oint_{\Gamma(\omega_0)} \Xi(r,\Omega)e^{-i\Omega t}\mathrm{d}\Omega, \qquad [7.35]$$

where $\Gamma(\omega_0)$ denotes a positively oriented path encircling ω_0.

Define a function $\mathscr{D}(\Omega)$ of complex frequency Ω by

$$\mathscr{D}(\Omega) = 2\pi\int_0^1 \overline{\Xi(r,\overline{\Omega})}\mathscr{E}(\Omega)\Xi(r,\Omega)\mathrm{d}r, \qquad [7.36]$$

in such a way that the wave action is obtained by the residue of $\mathscr{D}(\Omega)$ at $\Omega = \omega_0$ as follows [HIR 08a, HIR 09]

$$2\mu_0 = \frac{1}{2\pi i}\oint_{\Gamma(\omega_0)} \mathscr{D}(\Omega)\mathrm{d}\Omega. \qquad [7.37]$$

By expanding $\mathscr{E}(\Omega)$ in [7.36], in the neighborhood of $\Omega = \omega_0 (\in \mathbb{R})$, as $\mathscr{E}(\Omega) = \mathscr{E}(\omega_0) + (\Omega-\omega_0)\partial\mathscr{E}(\omega_0)/\partial\Omega + \ldots$, we obtain from [7.37], with use of $\mathscr{E}(\omega_0)r\hat{\xi}_r = 0$,

$$2\mu_0 = 2\pi\int_0^1 r\hat{\xi}_r\frac{\partial\mathscr{E}}{\partial\Omega}(\omega_0)r\hat{\xi}_r\mathrm{d}r. \qquad [7.38]$$

We can easily confirm that [7.38] indeed agrees with [7.33] because of

$$\frac{\partial\mathscr{E}}{\partial\Omega}(\omega_0) = \frac{2}{\omega_0-m}\mathscr{E}(\omega_0) - \frac{4k^2rm}{(m^2+k^2r^2)^2} + \frac{8k^2r}{(\omega_0-m)(m^2+k^2r^2)}. \qquad [7.39]$$

A further simplification of μ_0 is achieved for Kelvin waves. Let us introduce a function derived from [7.26],

$$\Phi(r,\Omega) = r\left[-(\Omega-m)\eta J_{m+1}(\eta r) + \frac{m}{r}(\Omega-m-2)J_m(\eta r)\right], \qquad [7.40]$$

where $\eta = \eta(\Omega; m, k_0)$ is defined by [7.27]. In view of [7.22], $\Phi(r,\Omega)$ is a solution of $\mathscr{E}(\Omega)\Phi(r,\Omega) = 0$ satisfying the boundary condition $\Phi(0,\Omega) = 0$ at $r = 0$, for arbitrary $\Omega \in \mathbb{C}$. The dispersion relation [7.28] is recovered by imposing $\Phi(1,\omega_0) = 0$.

We separate a singular term at $\Omega = \omega_0$ in $\Xi(r,\Omega)$ as

$$\Xi(r,\Omega) = \frac{i}{(\Omega-\omega_0)}\frac{\Omega-m}{\sqrt{4-(\Omega-m)^2}}\Phi(r,\Omega) + \Psi(r,\Omega), \qquad [7.41]$$

where

$$\Psi(r,\Omega) = \frac{i}{\Omega-\omega_0}\left[r\hat{\xi}_r(r;\omega_0) - \frac{\Omega-m}{\sqrt{4-(\Omega-m)^2}}\Phi(r,\Omega)\right]. \qquad [7.42]$$

The second term $\Psi(r,\Omega)$ is regular at $\Omega = \omega_0$ whereas the first term contains a pole $(\Omega-\omega_0)^{-1}$. The boundary conditions read $\Xi(0,\Omega) = \Psi(0,\Omega) = 0$ and $\Xi(1,\Omega) = 0$.

Substituting [7.41] into [7.36] and performing partial integration twice for the differential operator $\mathscr{E}(\Omega)$ of the Sturm–Liouville type, we are left with, by virtue of $\mathscr{E}(\Omega)\Phi(r,\Omega) = 0$,

$$\begin{aligned}\mathscr{D}(\Omega) &= 2\pi\int_0^1 \left[\overline{\mathscr{E}(\overline{\Omega})\Psi(r,\overline{\Omega})}\right]\Psi(r,\Omega)\mathrm{d}r \\ &\quad -\frac{2\pi(\Omega-m)^4}{(\Omega-\omega_0)^2[4-(\Omega-m)^2](m^2+k^2)}\frac{d\Phi}{dr}(1,\Omega)\Phi(1,\Omega) \\ &\quad +2\pi\frac{(\Omega-m)^2}{m^2+k^2}\overline{\frac{\mathrm{d}\Psi}{\mathrm{d}r}(1,\overline{\Omega})}\Psi(1,\Omega). \end{aligned} \qquad [7.43]$$

Note that a singularity at $\Omega = \omega_0$ resides only in the second term. The regular terms have no contribution for the calculation of the residue [7.37]. The dispersion relation $\Phi(1,\Omega)$ has the property $\Phi(1,\Omega) \propto (\Omega-\omega_0)$ in the neighborhood of $\Omega = \omega_0$, and it can be shown that $\mathrm{d}\Phi(1,\Omega)/\mathrm{d}r$ never vanishes at $\Omega = \omega_0$. For a Kelvin wave, we can directly deduce the derivative of [7.40] at $r = 1$ as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r}(1,\Omega) = -\left(m^2+k^2\right)\frac{4-(\Omega-m)^2}{\Omega-m}J_m(\eta) - \frac{2m}{\Omega-m}\Phi(1,\Omega). \qquad [7.44]$$

The second term $\propto (\Omega - \omega_0)$ on the right-hand side provides a regular term in $\mathscr{D}(\Omega)$ and therefore the singular part of [7.43] simplifies to

$$\mathscr{D}(\Omega) = \frac{2\pi}{(\Omega-\omega_0)^2}D(\Omega;m,k)+\ldots, \qquad [7.45]$$

where

$$D(\Omega;m,k) = (\Omega-m)^3 J_m(\eta)\Phi(1,\Omega). \qquad [7.46]$$

The dispersion relation $\omega_0 = \omega_0(m,k_0)$ is provided by $D(\omega_0,m,k_0) = 0$. With this form, the wave action [7.37] gives rise to a desired formula of the wave action written in terms of a derivative of the dispersion relation

$$2\mu_0 = 2\pi\frac{\partial D}{\partial\Omega}(\omega_0;m,k), \qquad [7.47]$$

completing the proof of [7.4] for the wave energy $E = \omega_0\mu_0$.

Substituting from [7.40], [7.47] becomes

$$\mu_0 = 2\pi(\omega_0-m)\left[m(\omega_0-m)+2(m^2+k^2)\right]J_m^2(\eta_0). \qquad [7.48]$$

It should be considered that the coefficient depends on the normalization [7.26] for ξ_r. The representation [7.48] of the action, multiplied by square of amplitude constant, is directly worked out from [7.17] [MIE 10].

7.5. Conclusion

We have systematically demonstrated, in the context of Kelvin waves, how the wave action is associated with a derivative of the dispersion relation with respect to the frequency, being akin to the dielectric energy. Our derivation presents the close relation between the two, and could conceivably be applicable to various waves described by the eigenvalue problem of the Sturm–Liouville type. We remark that formula [7.38] holds true for cases where the eigenvalue problem does not have an analytical solution but can only be solved numerically.

As a by-product of using the Lagrangian displacement, an appropriate average of [7.13] provides us with a mean flow induced by quadratic interaction of Kelvin waves in an unambiguous way [FUK 08]

$$\overline{\boldsymbol{v}_2} = \overline{\mathscr{P}\left[\boldsymbol{\xi}_1 \times \left(\nabla \times (\boldsymbol{\xi}_1 \times \boldsymbol{\omega}_0)\right) + \boldsymbol{\xi}_2 \times \boldsymbol{\omega}_0\right]}. \qquad [7.49]$$

We have revealed that the mean flow has a close bearing with the pseudo-momentum [FUK 10, FUK 13], in common with the generalized Lagrangian mean (GLM) theory [AND 78, BÜH 09]. Using the obtained mean flow of $O(\alpha^2)$, we can derive the weakly nonlinear amplitude equations to $O(\alpha^3)$ [MIE 10, FUK 13]. Not only the Hamiltonian pitchfork bifurcation [GUC 92] but also the Hamiltonian Hopf bifurcation [KNO 94] is tractable. Notably, with an appropriate normalization of the amplitude variables, the amplitude equations are shown to be reducible to canonical Hamiltonian equations [FUK 13]. This fact provides some support for our general framework of using the Lagrangian variable for weakly nonlinear stability analysis of the MSTW instability. Our hybrid method of incorporating the Lagrangian treatment into the Eulerian framework completes the determination of all the coefficients by advancing the previous treatment [SIP 00]. Details will be reported elsewhere.

7.6. Bibliography

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