Chapter 3

The Sign Exchange Bifurcation in a Family of Linear Hamiltonian Systems

This chapter discusses an example of a one-parameter family of Hamiltonian systems [SAD 99] in detail, which describe the behavior of two classical particles modeling angular momenta that interact in a magnetic field. This system exhibits an S^1-equivariant sign exchange bifurcation in its linearization about an equilibrium point [SAD 10]. The stability of this bifurcation under small S^1-invariant perturbations by linear Hamiltonian vector fields is shown: at bifurcation points, the family is S^1-equivariantly versal. The versality is shown by calculating a transverse slice in the set of liner S^1-equivariant infinitesimally real symplectic matrices to the S^1-equivariant real symplectic orbit of the infinitesimally symplectic matrix in the family at the bifurcation point. The method we use to compute this slice is different from the centralizer approach described in [MEL 93a] and [MEL 93b]. In [SAD 99], the authors show that as the parameter varies, two of the quantum states of the semi-classical limit cross each other at a unique parameter value. This crossing amounts to a redistribution of the quantum states as the parameter varies.

3.1. Statement of problem

We now describe the specific one-parameter family of Hamiltonian systems.

Phase space: phase space for the system is $\mathbb{R}^3 \times \mathbb{R}^3$ with coordinates $(S_x, S_y, S_z, L_x, L_y, L_z)$. Put a Euclidean inner product $\cdot$ on $\mathbb{R}^3$ so that $S \cdot S = S_x^2 + S_y^2 + S_z^2$, $L \cdot L =$

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$L_x^2+L_y^2+L_z^2$, and $S\cdot L=S_xL_x+S_yL_y+S_zL_z$. Also, let $s>0$ and $\ell>0$ be two fixed values that define the spheres $s^2=S\cdot S$ and $\ell^2=L\cdot L$, respectively.

Poisson bracket: on $C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$, the structure matrix of the Poisson bracket $\{\ ,\ \}$ at $(S,L)\in\mathbb{R}^3\times\mathbb{R}^3$ is the 6×6 skew symmetric matrix $\mathscr{W}(S,L)=\begin{pmatrix}\widehat{S} & 0\\ 0 & \widehat{L}\end{pmatrix}$, where $\widehat{x}=\begin{pmatrix}0 & x_3 & -x_2\\ -x_3 & 0 & x_1\\ x_2 & -x_1 & 0\end{pmatrix}$ if $x=(x_1,x_2,x_3)$.

Hamiltonian: for each $\gamma\in[0,1]$, the Hamiltonian is

$$H_\gamma:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}:(S,L)\mapsto \tfrac{1-\gamma}{s}S_z+\tfrac{\gamma}{s\ell}S\cdot L. \qquad [3.1]$$

Equations of motion: the motion of the system $(H,\mathbb{R}^3\times\mathbb{R}^3,\{\ ,\ \})$ is governed by the Hamiltonian vector field X_{H_γ} on $\mathbb{R}^3\times\mathbb{R}^3$, whose integral curves satisfy

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\{S,H_\gamma\}=\tfrac{1-\gamma}{s}e_3\times S-\tfrac{\gamma}{s\ell}S\times L \qquad [3.2a]$$

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\{L,H_\gamma\}=\tfrac{\gamma}{s\ell}S\times L. \qquad [3.2b]$$

Here, $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{R}^3$.

Integrals: the functions

$$C_1=S\cdot S=S_x^2+S_y^2+S_z^2 \quad\text{and}\quad C_2=L\cdot L=L_x^2+L_y^2+L_z^2$$

are integrals of the vector field X_{H_γ} because they are Casimirs of the Poisson algebra $(C^\infty(\mathbb{R}^3\times\mathbb{R}^3),\{\ ,\ \},*)$. Here, $*$ is pointwise multiplication of functions. Thus, the smooth manifold $S_s^2\times S_\ell^2$ defined by $C_1=s^2$ and $C_2=\ell^2$ is invariant under the flow of X_{H_γ}.

S^1 *symmetry*: consider the vector field X_{J_z} on $\mathbb{R}^3\times\mathbb{R}^3$, where $J_z=S_z+L_z$. Because the integral curves of X_{J_z} satisfy

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\{S,J_z\}=e_3\times S \quad\text{and}\quad \frac{\mathrm{d}L}{\mathrm{d}t}=\{L,J_z\}=e_3\times L,$$

the flow $\varphi_t^{J_z}$ of X_{J_z} is given by

$$\mathbb{R}\times(\mathbb{R}^3\times\mathbb{R}^3)\to\mathbb{R}^3\times\mathbb{R}^3:\big(t,\begin{pmatrix}S\\L\end{pmatrix}\big)\mapsto \mathrm{diag}(R_t,1,R_t,1)\begin{pmatrix}S\\L\end{pmatrix},$$

where $R_t = \left(\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right)$. Since $\varphi_t^{J_z}$ is periodic of period 2π, it defines an S^1-action on $\mathbb{R}^3 \times \mathbb{R}^3$. This S^1-action is a symmetry of the vector field X_{H_γ}, because the flow of X_{J_z} commutes with the flow of X_{H_γ}. This follows from $[X_{H_\gamma}, X_{J_z}] = X_{H_\gamma}X_{J_z} - X_{J_z}X_{H_\gamma} = 0$. Since $\varphi_t^{J_z}$ leaves the functions C_1 and C_2 invariant, the vector field X_{J_z} restricts to a vector field on $S_s^2 \times S_\ell^2$, whose flow is the restriction of $\varphi_t^{J_z}$ to $S_s^2 \times S_\ell^2$.

Equilibrium points: the point $p = (S, L) \in \mathbb{R}^3 \times \mathbb{R}^3$ is an equilibrium point of X_{H_γ} if and only if $X_{H_\gamma}(p) = 0$, that is if and only if

$$\tfrac{1-\gamma}{s}\, e_3 \times S = 0 \qquad [3.3a]$$

$$\tfrac{\gamma}{s\ell}\, S \times L = 0. \qquad [3.3b]$$

Equation [3.3b] implies that the vectors S and L are collinear, while equation [3.3a] implies that S, and hence L, is collinear with e_3. Thus, $p = p_{\eta_s,\eta_\ell} = (\eta_s s\, e_3, \eta_\ell \ell\, e_3)$, where $\eta_s^2 = \eta_\ell^2 = 1$, are the only equilibrium points of X_{H_γ}. Observe that p lies on the invariant manifold $S_s^2 \times S_\ell^2$.

Linearization of X_{H_γ} at p_{η_s,η_ℓ}: the linearization of X_{H_γ} at $p = p_{\eta_s,\eta_\ell}$ is $DX_{H_\gamma}(p)$, which is a linear vector field on $T_p(S_s^2 \times S_\ell^2)$, the tangent space of $S_s^2 \times S_\ell^2$ at p. Now

$$\begin{aligned} T_p(S_s^2 \times S_\ell^2) &= \ker \begin{pmatrix} S_x & S_y & S_z & 0 & 0 & 0 \\ 0 & 0 & 0 & L_x & L_y & L_z \end{pmatrix}\Bigg|_{p=p_{\eta_s,\eta_\ell}} \\ &= \ker \begin{pmatrix} 0 & 0 & \eta_s s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_\ell \ell \end{pmatrix} = \operatorname{span}\{e_1, e_2, e_4, e_5\} = \mathbb{R}^4, \end{aligned}$$

where $\{e_1, \ldots, e_6\}$ is the standard basis of $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$. From now on, we use $\{e_1, e_2, e_3 e_4\}$ as the standard basis for $\mathbb{R}^4$ instead of $\{e_1, e_2, e_4, e_5\}$. Using the notation $(\delta S, \delta L)$ to represent a derivation at p_{η_s,η_ℓ} on $C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, that is a tangent vector to $\mathbb{R}^3 \times \mathbb{R}^3$ at $p = p_{\eta_s,\eta_\ell}$, we obtain

$$\frac{\mathrm{d}(\delta S)}{\mathrm{d}t} = \tfrac{1-\gamma}{s}\, e_3 \times \delta S - \frac{\mathrm{d}(\delta L)}{\mathrm{d}t}$$

$$\frac{\mathrm{d}(\delta L)}{\mathrm{d}t} = \tfrac{\gamma}{s\ell}\Big(\delta S \times L + S \times \delta L\Big)\Big|_{p=p_{\eta_s,\eta_\ell}} = \tfrac{\gamma}{s\ell}\big(\delta S \times (\eta_\ell \ell e_3) + (\eta_s s e_3) \times \delta L\big).$$

In other words, $\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta S \\ \delta L\end{pmatrix} = A_\gamma \begin{pmatrix}\delta S \\ \delta L\end{pmatrix}$, where $A_\gamma = \left(\begin{array}{ccc|ccc} 0 & -\alpha & 0 & 0 & \mu & 0 \\ \alpha & 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \nu & 0 & 0 & -\mu & 0 \\ -\nu & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ and

$$\alpha = \tfrac{1}{s}\big((1-\gamma) + \eta_\ell \gamma\big), \quad \mu = \tfrac{\eta_s}{\ell}\gamma \quad \text{and} \quad \nu = \tfrac{\eta_\ell}{s}\gamma. \qquad [3.4]$$

Thus, the linearization $DX_{H_\gamma}(p)$ of X_{H_γ} at $p = p_{\eta_s,\eta_\ell}$ is

$$X_\gamma = A_\gamma \big| T_p(S_s^2 \times S_\ell^2) = \begin{pmatrix} 0 & -\alpha & 0 & \mu \\ \alpha & 0 & -\mu & 0 \\ 0 & \nu & 0 & -\mu \\ -\nu & 0 & \mu & 0 \end{pmatrix}. \qquad [3.5]$$

Symplectic form on $T_p(S_s^2 \times S_\ell^2)$: the structure matrix $\mathscr{W}(p)$ of the Poisson bracket $\{\ ,\ \}$ at $p = p_{\eta_s,\eta_\ell}$ is $\begin{pmatrix} \eta_s s\widehat{e}_3 & 0 \\ 0 & \eta_\ell \ell \widehat{e}_3 \end{pmatrix}$. Thus, we obtain $W = \mathscr{W}(p) \big| T_p(S_s^2 \times S_\ell^2) = \begin{pmatrix} 0 & \eta_s s & 0 & 0 \\ -\eta_s s & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_\ell \ell \\ 0 & 0 & -\eta_\ell \ell & 0 \end{pmatrix}$, which yields the symplectic form on $T_p(S_s^2 \times S_\ell^2)$ given by

$$\Omega = (W^{-1})^T = \begin{pmatrix} 0 & \frac{\eta_s}{s} & 0 & 0 \\ -\frac{\eta_s}{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\eta_\ell}{\ell} \\ 0 & 0 & -\frac{\eta_\ell}{\ell} & 0 \end{pmatrix}. \qquad [3.6]$$

A short calculation shows that X_γ is Ω-infinitesimally symplectic, that is $X \in \mathrm{sp}(4,\Omega)$. In other words, $X_\gamma^T \Omega + \Omega X_\gamma = 0$.

From now on, we assume that $s = \varepsilon^2 \ell$, where $0 < \varepsilon \ll 1$. The main goal of this chapter is to show that as γ increases, the S^1-invariant curve

$$\mathscr{X} : [0,1] \to \mathrm{sp}(4,\Omega) : \gamma \mapsto X_\gamma = DX_{H_\gamma}(p) \qquad [3.7]$$

at $p = p_{1,-1}$ undergoes the following S^1-equivariant bifurcations at those values of γ where X_γ has multiple eigenvalues, which are near $\frac{1}{2}$ because of the scaling. Specifically:

1) At $\gamma_- = \frac{1}{2+2\varepsilon+\varepsilon^2}$, the curve X_γ has a linear Hamiltonian Hopf bifurcation with symplectic sign $\rho_- = -1$.

2) At $\gamma^+ = \frac{1}{2-\varepsilon^2}$, the curve X_γ has a switch twist bifurcation, where the sense of the rotational part of X_γ reverses.

3) At $\gamma_+ = \frac{1}{2-2\varepsilon+\varepsilon^2}$, the curve X_γ has a linear Hamiltonian Hopf bifurcation with symplectic sign $\rho_+ = 1$.

4) When γ moves from $[0, \gamma_-)$ to $(\gamma_+, 1]$, the infinitesimally symplectic matrix X_γ undergoes a global S^1-equivariant sign exchange bifurcation, where the symplectic signs of its purely imaginary eigenvalues, listed in order of increasing absolute value, are interchanged.

At $p = p_{-1,-1}$, the curve $\mathscr{X}$ [3.7] has a sign exchange bifurcation at γ^-.

All of the above bifurcations are $\mathrm{sp}(4,\mathbb{R})$-versal under the S^1 symmetry. Thus, the curve $\mathscr{X}$ is S^1-equivariantly $\mathrm{sp}(4,\mathbb{R})$-typical.

3.2. Bifurcation values of γ

To determine when the family $\mathscr{X}$ [3.7] has bifurcations, that is when X_γ [3.5] has multiple eigenvalues, we compute the characteristic polynomial χ_{X_γ} of X_γ and then determine when its discriminant Δ vanishes.

By definition, the characteristic polynomial of X_γ is

$$\begin{aligned}\chi_{X_\gamma}(x) &= \det(X_\gamma - xI) = x^4 + (\mu^2 + 2\mu\nu + \alpha^2)x^2 + \mu^2(\alpha-\nu)^2 \\ &= \big((x^2 + i(\mu+\alpha)x + \mu(\nu-\alpha)\big)\big((x^2 - i(\mu+\alpha)x + \mu(\nu-\alpha)\big),\end{aligned} \quad [3.8]$$

whose discriminant, when χ_{X_γ} is thought of as a quadratic polynomial [3.8] in x^2, is

$$\Delta = (\mu^2 + 2\mu\nu + \alpha^2)^2 - 4\mu^2(\alpha-\nu)^2 = (\mu+\alpha)^2\widehat{\Delta}, \quad [3.9]$$

where $\widehat{\Delta} = (\mu-\alpha)^2 + 4\mu\nu$. Then, χ_{X_γ} has multiple roots if and only if $\Delta = 0$, that is, either

$$0 = \mu + \alpha = \tfrac{1}{s\gamma^0}(\gamma^0 - \gamma) \quad \text{where} \quad \gamma^0 = \frac{\ell}{\ell(1-\eta_\ell) - s\eta_s} \quad [3.10]$$

or

$$0 = \widehat{\Delta} = \Big(\tfrac{1}{s}\big((1-\gamma) + \eta_\ell\gamma\big) - \tfrac{\eta_s}{\ell}\gamma\Big)^2 + \tfrac{4\eta_s\eta_\ell}{s\ell}\gamma^2. \quad [3.11]$$

Recall that we have assumed that $s = \varepsilon^2\ell$, where $0 < \varepsilon \ll 1$. Suppose that $\mu + \alpha = 0$. If $\eta_\ell = 1$, then $\gamma^0 = \frac{-1}{\varepsilon^2\eta_s}$ is < 0 if $\eta_s = 1$, or is > 1 if $\eta_s = -1$ when $0 < \varepsilon \ll 1$. Both of these conclusions contradict the fact that $0 \le \gamma \le 1$. Therefore, $\eta_\ell = -1$. So, $\mu + \alpha = 0$ if and only if $\eta_s = \pm 1$, $\eta_\ell = -1$ and

$$\gamma^{\pm} = \gamma^0 = \frac{1}{2 \mp \varepsilon^2} = \tfrac{1}{2} \pm \tfrac{1}{2}\varepsilon^2 + \mathrm{O}(\varepsilon^4). \quad [3.12]$$

Now suppose that $\widehat{\Delta} = 0$. If $\eta_s \eta_\ell = 1$, then from [3.11] it follows that $\widehat{\Delta} > 0$. But this contradicts the hypothesis that $\widehat{\Delta} = 0$. Therefore, $\eta_s \eta_\ell = -1$. If $\eta_s = -1$ and $\eta_\ell = 1$, then

$$0 = \widehat{\Delta} = \left(\tfrac{1}{s} + (\tfrac{1}{\ell} + \tfrac{2}{\sqrt{s\ell}})\gamma\right)\left(\tfrac{1}{s} + (\tfrac{1}{\ell} - \tfrac{2}{\sqrt{s\ell}}\gamma)\right) = \left(\gamma + \tfrac{1}{\varepsilon}\frac{1}{2+\varepsilon}\right)\left(\gamma - \tfrac{1}{\varepsilon}\frac{1}{2-\varepsilon}\right),$$

which solved for γ, contradict the fact that $\gamma \in [0,1]$ when $0 < \varepsilon \ll 1$. Therefore, $\eta_s = 1$ and $\eta_\ell = -1$. So, $0 = \widehat{\Delta} = \frac{1}{s^2 \gamma_+ \gamma_-}(\gamma - \gamma_-)(\gamma - \gamma_+)$ if and only if $\eta_s = 1$ and $\eta_\ell = -1$ and

$$\gamma_\pm = \frac{\ell}{2\ell + s \mp 2\sqrt{s\ell}} = \frac{1}{2 \mp 2\varepsilon + \varepsilon^2} = \tfrac{1}{2} \pm \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2 + \mathrm{O}(\varepsilon^4). \qquad [3.13]$$

Thus, we have shown the following.

CLAIM 2.1.– When $p = p_{1,-1}$, the smooth family $\mathscr{X} : [0,1] \to \mathrm{sp}(4,\Omega) : \gamma \mapsto X_\gamma$ has bifurcations only when γ equals $\gamma_\pm$ or γ^+, while when $p = p_{-1,-1}$, they occur only when $\gamma = \gamma^-$.

3.3. Versal normal forms near the bifurcation values

In this section, we find a smooth normal form for X_γ near the bifurcation values of γ and show that the S^1-invariant curve $\mathscr{X}$ [3.7] is versal and hence is typical.

3.3.1. *Normal forms*

We start by giving the infinitesimally symplectic normal form of X_γ.

CLAIM 3.1.– The first two entries in Table 3.1 give the infinitesimally symplectic normal form of X_γ at $p = p_{1,-1}$ for the bifurcation values of γ and the third entry gives the normal form at the bifurcation values when $p = p_{-1,-1}$.

With respect to the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ such that the infinitesimally symplectic matrix X_γ is in normal form, the matrix of the symplectic form Ω on $\mathbb{R}^4$ is $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Conditions	Normal form of X_γ	Remarks
1. $\gamma = \gamma^+ = \frac{1}{2-\varepsilon^2}$, $0 < \varepsilon \ll 1$	$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$	$\lambda > 0$, see [3.33].
2. $\gamma = \gamma_\pm = \frac{1}{2\mp 2\varepsilon+\varepsilon^2}$, $0 < \varepsilon \ll 1$	$\begin{pmatrix} 0 & \mp\beta_\pm & 0 & 0 \\ \pm\beta_\pm & 0 & 0 & 0 \\ \rho_\pm & 0 & 0 & \mp\beta_\pm \\ 0 & \rho_\pm & \pm\beta_\pm & 0 \end{pmatrix}$	$\mp\beta_\pm > 0$, see [3.35]; $\rho_\pm = \pm 1$, see [3.37]
3. $\gamma = \gamma^- = \frac{1}{2+\varepsilon^2}$, $0 < \varepsilon \ll 1$	$\begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ -\beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}$	$\beta > 0$, see [3.34]. the symplectic signs are $-$, $+$

Table 3.1. *Infinitesimally symplectic normal form of X_γ at the bifurcation values of γ*

3.3.2. *Linear Hamiltonian Hopf bifurcation at* $\gamma_\pm$

Here, we show that the smooth curve $\gamma \mapsto X_\gamma$ undergoes an S^1-equivariant linear Hamiltonian Hopf bifurcation when γ traverses an open interval containing $\gamma_\pm$.

Recall some basic facts about the *linear Hamiltonian Hopf bifurcation* [CUS 97]. Consider the 4×4 real infinitesimally symplectic matrix $Y_0 = \begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \rho & 0 & 0 & -\beta \\ 0 & \rho & \beta & 0 \end{pmatrix}$ on $(\mathbb{R}^4, \omega)$, where $\beta > 0$, $\rho^2 = 1$ and $\omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then, Y_0 is in infinitesimally symplectic normal form. The $\mathrm{Sp}(4,\mathbb{R})$-adjoint orbit $\mathscr{O}_{Y_0} = \{P^{-1}Y_0P \in \mathrm{sp}(4,\mathbb{R})$ $P \in \mathrm{Sp}(4,\mathbb{R})\}$ through Y_0 is a smooth manifold whose tangent space at Y_0 is the image of the mapping

$$\mathrm{ad}_{Y_0} : \mathrm{sp}(4,\mathbb{R}) \to \mathrm{sp}(4,\mathbb{R}) : X \mapsto [Y_0, X] = Y_0X - XY_0.$$

LEMMA 3.1–

$$\mathrm{sp}(4,\mathbb{R}) = \mathrm{im}\,\mathrm{ad}_{Y_0} \oplus \mathrm{span}\{S, M\}, \qquad [3.14]$$

where $S = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

PROOF.– For more details, see [CUS 97, p. 259].

CLAIM 3.2.– Suppose that $\gamma : \mathbb{R} \to \mathrm{sp}(4,\mathbb{R}) : \mu \mapsto Y_\mu$ is a smooth family of real infinitesimally symplectic matrices on $(\mathbb{R}^4, \omega)$ with Y_0 as given above. For every μ in some open interval I containing 0, there is a smooth family $P : I \to \mathrm{Sp}(4,\mathbb{R}) : \mu \mapsto P_\mu$

with $P_0 = I$ of real symplectic linear mappings, which transforms γ into the smooth normal form $\Gamma : I \to \mathrm{sp}(4,\mathbb{R}) : \mu \mapsto Z_\mu = P_\mu^{-1} Y_\mu P_\mu$, where

$$Z_\mu = \begin{pmatrix} 0 & -(\beta+\nu_1(\mu)) & \nu_2(\mu) & 0 \\ \beta+\nu_1(\mu) & 0 & 0 & \nu_2(\mu) \\ \rho & 0 & 0 & -(\beta+\nu_1(\mu)) \\ 0 & \rho & \beta+\nu_1(\mu) & 0 \end{pmatrix}. \qquad [3.15]$$

The functions $\nu_i : I \to \mathbb{R} : \mu \mapsto \nu_i(\mu)$ are smooth with $\nu_i(0) = 0$.

PROOF.– Use the implicit function theorem. For more details, see [CUS 97, p. 258].

COROLLARY 3.1.– If either $\frac{d\nu_1}{d\mu} \neq 0$ or $\frac{d\nu_2}{d\mu} \neq 0$ at $\mu = 0$, then the curve $\mu \mapsto Y_\mu$ is transverse to the $\mathrm{Sp}(4,\mathbb{R})$-adjoint orbit $\mathscr{O}_{Y_0}$ at Y_0.

PROOF.– Since either $\frac{d\nu_1}{d\mu} \neq 0$ or $\frac{d\nu_2}{d\mu} \neq 0$, it follows that the smooth normal form curve $\mu \mapsto Z_\mu$ crosses the $\mathrm{Sp}(4,\mathbb{R})$-adjoint orbit $\mathscr{O}_{Z_0}$ transversely at Z_0, because the affine two-plane $Z_0 + \mathrm{span}\{S, M\}$ intersects $\mathscr{O}_{Z_0}$ transversely at Z_0. The corollary follows by applying the smooth symplectic change of coordinates P_μ to Z_μ to obtain Y_μ.

We can improve the result of corollary 3.1. Let $Y^\alpha = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \rho_\pm & 0 & 0 & -\alpha \\ 0 & \rho_\pm & \alpha & 0 \end{pmatrix} \in \mathrm{sp}(4,\mathbb{R})$, where $\alpha \neq 0$ and $\rho_\pm^2 = 1$. Let $\mathscr{S} = \{PY^\alpha P^{-1} \in \mathrm{sp}(4,\mathbb{R}) \,|\, P \in \mathrm{Sp}(4,\mathbb{R})\}$. Then, $\mathscr{S}$ is the set of all real infinitesimally symplectic 4×4 matrices with the same *symplectic shape* as $Y^\beta = Y_0$. Note that $\mathscr{S}$ is the disjoint union of $\mathrm{Sp}(4,\mathbb{R})$-adjoint orbits $\mathscr{O}_{Y^\alpha}$. Consider the subset $\mathscr{S}_0$ of $\mathscr{S}$, where $|\alpha - \beta|$ is small. Then, $\mathscr{S}_0$ is an open subset of $\mathscr{S}$, which is a smooth codimension 1 submanifold of $\mathrm{sp}(4,\mathbb{R})$. Since the curve $\mu \mapsto Y_\mu$ is transverse to $\mathscr{O}_{Y^\beta}$ at Y_0, from the transversality theorem it follows that any smooth curve of infinitesimally symplectic 4×4 matrices, which is sufficiently close to $\mu \mapsto Y_\mu$, intersects $\mathscr{S}_0$ transversely. In other words, we have proved the following.

COROLLARY 3.2.– Under the hypotheses of corollary 3.1, the curve $\gamma : \mathbb{R} \to \mathrm{sp}(4,\mathbb{R}) : \mu \to Y_\mu$ is $\mathrm{sp}(4,\mathbb{R})$-typical near 0.

In other words, the linear Hamiltonian Hopf bifurcation of the family $\mu \to Y_\mu$ can not be removed by a small $\mathrm{sp}(4,\mathbb{R})$ perturbation.

We now apply the above results to the smooth family $\gamma \mapsto X_\gamma$ when γ lies in some open neighborhood of $\gamma_\pm$ [3.13]. Here, $X_{\gamma_\pm}$ is given by second entry in Table 3.1. From [3.8], we see that the characteristic polynomial of X_γ is

$$x^4 + (\mu^2 + 2\mu\nu + \alpha^2)x^2 + \mu^2(\alpha - \nu)^2.$$

The characteristic polynomial of the smooth normal form Z_μ [3.15] of X_γ with $\beta = \pm\beta_\pm$ and $\rho = \rho_\pm$ is

$$x^4 + 2\big((\pm\beta_\pm + \nu_1)^2 - \rho_\pm \nu_2\big)x^2 + \big((\pm\beta_\pm + \nu_2)^2 + \rho_\pm \nu_2\big)^2.$$

Since X_γ and Z_μ are conjugate, their characteristic polynomials are equal, that is

$$\begin{aligned}(\pm\beta_\pm + \nu_1)^2 + \rho_\pm \nu_2 &= \mu(\alpha - \nu) > 0 \text{ and } (\pm\beta_\pm + \nu_1)^2 - \rho_\pm \nu_2 \\ &= \tfrac{1}{2}\,(\mu^2 + 2\mu\nu + \alpha^2).\end{aligned}$$

Solving the above equation gives

$$\nu_1(\gamma) = \mp\beta_\pm \pm \tfrac{1}{2}\,(\mu + \alpha) \text{ and } \nu_2(\gamma) = -\rho_\pm\big(\mu\nu + \tfrac{1}{4}(\mu - \alpha)^2\big) = -\tfrac{1}{4}\rho_\pm \widehat{\Delta}.$$

Since $\beta_\pm = \frac{1}{2}\,(\mu + \alpha)$ when $\gamma = \gamma_\pm$, we get $\nu_1(\gamma_\pm) = 0$. Since $\widehat{\Delta} = 0$ when $\gamma = \gamma_\pm$, we see that $\nu_2(\gamma_\pm) = 0$. Thus, the smooth normal form of X_γ, when γ is near $\gamma_\pm$, is $Z_\gamma = X_{\gamma_\pm} + \nu_1(\gamma)S + \nu_2(\gamma)M$. Because

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big|_{\gamma=\gamma_\pm} \nu_1(\gamma) = \frac{\mathrm{d}}{\mathrm{d}\gamma}\Big|_{\gamma=\gamma_\pm} \big(\mp\beta + \tfrac{1}{2}\,(\mu + \alpha)\big) = \tfrac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big|_{\gamma=\gamma_\pm} \big(\tfrac{1}{\ell}\gamma + \tfrac{1}{s}(1 - 2\gamma)\big) = \tfrac{1}{2\ell} - \tfrac{1}{\varepsilon^2\ell} \neq 0,$$

when $0 < \varepsilon \ll 1$, we can apply corollary 3.2. This shows that the curve $\gamma \mapsto X_\gamma$ crosses the $\mathrm{Sp}(4,\mathbb{R})$-adjoint orbit $\mathscr{O}_{X_{\gamma_\pm}}$ transversely at $X_{\gamma_\pm}$. From corollary 3.2, it follows that the curve $\gamma \mapsto X_\gamma$ is $\mathrm{sp}(4,\mathbb{R})$-typical near $\gamma_\pm$.

We now look for an equivariant version of claim 3.2 and its corollaries. Consider the abelian subgroup S^1 of $\mathrm{Sp}(4,\mathbb{R})$ generated by the infinitesimally symplectic map $X = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then, every $S_\theta = \begin{pmatrix} R_\theta & 0 \\ 0 & R_\theta \end{pmatrix}$ in S^1, where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, commutes with $Z_{\gamma_\pm}$, because it commutes with $S = \begin{pmatrix} 0 & -\beta_\pm & 0 & 0 \\ \beta_\pm & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_\pm \\ 0 & 0 & \beta_\pm & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. In other words, the real infinitesimally symplectic matrix $Z_{\gamma_\pm}$ is S^1-equivariant, that is $Z_{\gamma_\pm} \in \mathrm{sp}(4,\mathbb{R})^{S^1}$.

Next we show how to obtain the smooth S^1-equivariant normal form of X_γ for γ near $\gamma_\pm$. We average. In more detail, let $\overline{P}_\gamma = \frac{1}{2\pi}\int_0^{2\pi} S_\theta^{-1} P_\gamma S_\theta \,\mathrm{d}\theta$, where $\gamma \mapsto P_\gamma$ is the smooth curve near $\gamma_\pm$ of symplectic matrices, which brings X_γ into smooth normal form, that is, $P_\gamma^{-1} X_\gamma P_\gamma = Z_\gamma$. Clearly, $\overline{P}_\gamma$ is linear and the mapping $\gamma \mapsto \overline{P}_\gamma$ is smooth. The following argument shows that $\overline{P}_\gamma \in \mathrm{Sp}(4,\mathbb{R})$. For $v, w \in \mathbb{R}^4$, we have

$$\omega(\overline{P}_\gamma v, \overline{P}_\gamma w) = \tfrac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\omega(S_\theta^{-1}P_\gamma S_\theta v, S_\psi^{-1}P_\gamma S_\psi w)\,\mathrm{d}\theta\,\mathrm{d}\psi$$
$$= \omega\Big(v, \tfrac{1}{2\pi}\int_0^{2\pi}\Big(S_\theta^{-1}P_\gamma^{-1}\Big(\tfrac{1}{2\pi}\int_\theta^{\theta-2\pi}S_\vartheta P_\gamma S_\vartheta^{-1}\,\mathrm{d}\vartheta\Big)S_\theta\,\mathrm{d}\theta\Big)w\Big),$$
where $\vartheta = \theta - \psi$, $S_\vartheta = S_\theta S_\psi^{-1}$ and S_θ, P_γ are symplectic
$$= \omega\Big(v, \Big(\tfrac{1}{2\pi}\int_0^{2\pi}S_\theta^{-1}P_\gamma^{-1}\overline{P}_\gamma S_\theta\,\mathrm{d}\theta\Big)w\Big) = \omega\Big(v, \Big(\tfrac{1}{2\pi}\int_0^{2\pi}S_\theta^{-1}P_\gamma^{-1}S_\theta\,\mathrm{d}\theta\Big)\overline{P}_\gamma w\Big)$$
$$= \omega(v, \overline{P}_\gamma^{-1}\overline{P}_\gamma w) = \omega(v,w).$$

The third to last equality above follows because

$$S_\theta^{-1}\overline{P}_\gamma S_\theta = \tfrac{1}{2\pi}\int_0^{2\pi}S_\theta^{-1}S_\psi^{-1}P_\gamma S_\psi S_\theta\,\mathrm{d}\psi = \tfrac{1}{2\pi}\int_\theta^{2\pi+\theta}S_\vartheta^{-1}P_\gamma S_\vartheta\,\mathrm{d}\vartheta = \overline{P}_\gamma.$$

The second equality above follows because $S_\psi S_\theta = S_{\psi+\theta} = S_\vartheta$. Thus, $\overline{P}_\gamma$ is S^1-equivariant, that is $\overline{P}_\gamma \in \mathrm{Sp}(4,\mathbb{R})^{S^1}$. Averaging the equation $S_\theta^{-1}Z_\gamma S_\theta = (S_\theta^{-1}P_\gamma S_\theta)^{-1}(S_\theta^{-1}X_\gamma S_\theta)(S_\theta^{-1}P_\gamma S_\theta)$ over $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ gives $Z_\gamma = \overline{P}_\gamma^{-1}X_\gamma\overline{P}_\gamma$, since X_γ and Z_γ are S^1-equivariant. Using corollary 3.2, we obtain the following.

CLAIM 3.3.– For all γ near $\gamma_\pm$, the smooth curve $\gamma \mapsto X_\gamma$ is $\mathrm{sp}(4,\mathbb{R})^{S^1}$-typical.

In other words, the S^1-equivariant linear Hamiltonian Hopf bifurcation at $\gamma_\pm$ cannot be removed by a small S^1-equivariant infinitesimally symplectic perturbation.

3.3.3. *The switch twist bifurcation at γ^+*

Here, we show that for all γ near γ^+, the smooth curve $\gamma \mapsto X_\gamma$ [3.5] undergoes an S^1-equivariant symplectic switch twist bifurcation.

Consider the real 4×4 infinitesimally symplectic matrix $Y_0 = \begin{pmatrix}\lambda I & 0\\ 0 & -\lambda I\end{pmatrix}$, where $\lambda > 0$. We look for an S^1-equivariant versal normal form. Recall that S^1 is the subgroup of $\mathrm{Sp}(4,R)$ consisting of all 4×4 matrices $S_\theta = \begin{pmatrix}R_\theta & 0\\ 0 & R_\theta\end{pmatrix}$, where $R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

LEMMA 3.2.–

$$\mathrm{sp}(4,\mathbb{R})^{S^1} = \left\{X = \begin{pmatrix}a & -b & c & 0\\ b & a & 0 & c\\ d & 0 & -a & -b\\ 0 & d & b & -a\end{pmatrix} \in \mathrm{sp}(4,\mathbb{R})\,\Big|\,(a,b,c,d)\in\mathbb{R}^4\right\}. \qquad [3.16]$$

PROOF.– Let $X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$ with $B^T = B$ and $C^T = C$. Then, $X \in \mathrm{sp}(4,\mathbb{R})$. The matrix X commutes with S_θ for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ if and only if A, B and C commute with R_θ. Now $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $(a,b,c,d) \in \mathbb{R}^4$ commutes with R_θ if and only if $c = -b$ and $a = d$. So, $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $(a,b) \in \mathbb{R}^2$. Also, $C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ commutes with R_θ if and only if $b = 0$ and $c = a$. So, $C = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where $a \in \mathbb{R}$. Thus, [3.16] holds.

We now prove the following.

LEMMA 3.3.– Let $Y_0 = \begin{pmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{pmatrix}$ with $\lambda > 0$. Then

$$\mathrm{sp}(4,\mathbb{R})^{S^1} = \mathrm{im}\,\mathrm{ad}_{Y_0} \oplus \mathrm{span}\{T, R\}, \qquad [3.17]$$

where $T = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $R = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

PROOF.– We calculate. For $X = \begin{pmatrix} K & cI \\ dI & -K^T \end{pmatrix}$, where $K = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, we have $\mathrm{ad}_{Y_0} X = \begin{pmatrix} 0 & 2\lambda c I \\ -2\lambda d I & 0 \end{pmatrix}$. Then

$$\begin{aligned} \mathrm{sp}(4,\mathbb{R})^{S^1} &= \left\{ X = \begin{pmatrix} K & cI \\ dI & -K^T \end{pmatrix} \in \mathrm{sp}(4,\mathbb{R}) \right\} \\ &= \left\{ \begin{pmatrix} 0 & cI \\ dI & 0 \end{pmatrix} \Big| (c,d) \in \mathbb{R}^2 \right\} \oplus \left\{ \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & 0 & b & -a \end{pmatrix} \Big| (a,b) \in \mathbb{R}^2 \right\} \\ &= \mathrm{im}\,\mathrm{ad}_{Y_0} \oplus \mathrm{span}\{T, R\}. \end{aligned}$$

We now prove a result on smooth S^1-equivariant normal form.

CLAIM 3.4.– Suppose that $\gamma : \mathbb{R} \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \mu \mapsto Y_\mu$ is a smooth mapping with Y_0 as given above. For every μ in some open interval I containing 0, there is a smooth function $P : I \to \mathrm{Sp}(4,\mathbb{R})^{S^1} : \mu \mapsto P_\mu$ with $P_0 = I$, which transforms γ into the smooth S^1-equivariant normal form $\Gamma : I \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \mu \mapsto Z_\mu = P_\mu^{-1} Y_\mu P_\mu$ where

$$\begin{pmatrix} \lambda + \nu_1(\mu) & -\nu_2(\mu) & 0 & 0 \\ \nu_2(\mu) & \lambda + \nu_1(\mu) & 0 & 0 \\ 0 & 0 & -(\lambda + \nu_1(\mu)) & -\nu_2(\mu) \\ 0 & 0 & \nu_2(\mu) & -(\lambda + \nu_1(\mu)) \end{pmatrix}. \qquad [3.18]$$

For $i = 1, 2$, the functions $\nu_i : I \to \mathbb{R} : \mu \mapsto \nu_i(\mu)$ are smooth and $\nu_i(0) = 0$.

PROOF.– Consider the mapping

$$\Phi : \mathbb{R}^2 \times \mathrm{Sp}(4,\mathbb{R})^{S^1} \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \mu \mapsto P(Y_0 + \nu_1 T + \nu_2 R)P^{-1}.$$

The rest of the argument follows using the implicit function theorem.

We can now explain what we mean by the switch twist bifurcation. From the smooth normal form [3.18] we see that the Lagrange planes $\Lambda_1 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$ and $\Lambda_2 = \mathrm{span}_{\mathbb{R}}\{e_3, e_4\}$ of $(\mathbb{R}^4, \omega)$ are Z_μ-invariant for all $\mu \in I$. They are also oriented by the two-forms $e_1 \wedge e_2$ and $e_3 \wedge e_4$, respectively. Thus, it makes sense to talk about clockwise and counterclockwise rotations on Λ_1 and Λ_2. We define the *rotational part* of $Z_\mu|\Lambda_{1,2}$ to be $R_\mu = \exp\begin{pmatrix} 0 & -\nu_2(\mu) \\ \nu_2(\mu) & 0 \end{pmatrix}$. We say that the *twist* of Z_μ is positive for all $\mu \in I \setminus \{0\}$ if R_μ is a counterclockwise rotation with respect to the orientation on Λ_1 or equivalently Λ_2; otherwise, it is negative. If we assume that $\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \neq 0$ at $\mu = 0$, then as $\mu \in I$ passes through 0, the sign of $\nu_2(\mu)$ changes because it vanishes only at $\mu = 0$. Thus, the twist of Z_μ switches sign as $\mu \in I$ passes through 0. This is the *switch twist* bifurcation at $\mu = 0$.

We now apply claim 3.4 to the smooth family $\gamma \mapsto X_\gamma$ [3.5] when γ is near γ^+. Here, X_{γ^+} is given by first entry in Table 3.1. The characteristic polynomial of X_γ is

$$x^4 + (\mu^2 + 2\mu\nu + \alpha^2)x^2 + \mu^2(\alpha - \nu)^2, \qquad [3.19]$$

using [3.8]. The characteristic polynomial of the smooth normal form Z_γ [3.18] is

$$x^4 + 2\big(-(\lambda + \nu_1)^2 + \nu_2^2\big)x^2 + \big((\lambda + \nu_1)^2 + \nu_2^2\big)^2. \qquad [3.20]$$

Since X_γ and Z_γ are conjugate, their characteristic polynomials [3.19] and [3.20] are equal, that is

$$(\lambda + \nu_1)^2 + \nu_2^2 = \mu(\alpha - \nu) \qquad [3.21a]$$

$$-(\lambda + \nu_1)^2 + \nu_2^2 = \tfrac{1}{2}(\mu^2 + 2\mu\nu + \alpha^2). \qquad [3.21b]$$

Solving [3.21a] and [3.21b] gives

$$\nu_2^2 = \tfrac{1}{2}(\mu + \alpha)^2 \qquad [3.22]$$

$$(\lambda + \nu_1)^2 = -\tfrac{1}{4}\big((\mu - \alpha)^2 + 4\mu\nu\big) = \tfrac{1}{4}(-\widehat{\Delta}). \qquad [3.23]$$

Since $\mu + \alpha = \frac{1}{s\gamma^+}(\gamma^+ - \gamma)$, from [3.22] it follows that

$$\nu_2(\gamma) = \tfrac{1}{\sqrt{2}}(\mu + \alpha). \qquad [3.24]$$

Hence, $\frac{dv_2}{d\gamma}$ at γ^+ equals $-\frac{1}{s\gamma^+} < 0$. Thus, $v_2(\gamma) > 0$, when $\gamma < \gamma^+$ and $v_2(\gamma) < 0$, when $\gamma > \gamma^+$. Thus, Z_γ undergoes a switch twist bifurcation at $\gamma = \gamma^+$. Because $-\widehat{\Delta} > 0$ for all γ near γ^+, using [3.23] we see that

$$v_1(\gamma) = -\lambda + \tfrac{1}{2}\sqrt{-(\mu+\alpha)^2 + 4\mu(\alpha - v)}. \qquad [3.25]$$

The plus sign is chosen in equation [3.25], because at γ^+ we have $\mu + \alpha = 0$ and $\lambda = \sqrt{-\alpha(\alpha - v)}$, which implies that $v_1(\gamma_+) = 0$. Thus, the smooth S^1-equivariant normal form of $\gamma \mapsto X_\gamma$ for all γ near γ^+ is $Z_\gamma = X_{\gamma^+} + v_1(\gamma)T + v_2(\gamma)R$. Since

$$\begin{aligned} \frac{d}{d\gamma}\Big|_{\gamma=\gamma^+} v_2(\gamma) &= \tfrac{1}{\sqrt{2}} \frac{d}{d\gamma}\Big|_{\gamma=\gamma^+} (\mu+\alpha) = \tfrac{1}{\sqrt{2}} \frac{d}{d\gamma}\Big|_{\gamma=\gamma^+} \big(\tfrac{2}{\ell}\gamma + \tfrac{1}{s}(1-2\gamma)\big) \\ &= \tfrac{1}{\sqrt{2}}\big(\tfrac{2}{\ell}\gamma^+ - \tfrac{2}{\ell\varepsilon^2}(2\gamma^+ - 1)\big) < 0, \end{aligned}$$

when $0 < \varepsilon \ll 1$. Using the argument in the proof of corollary 3.2, it follows that the smooth curve $\gamma \mapsto X_\gamma$ crosses the $\mathrm{Sp}(4,\mathbb{R})^{S^1}$-adjoint orbit $\mathscr{O}_{X_{\gamma^+}}$ transversely at X_{γ^+}. Let $\mathscr{S} = \{PX^\alpha P^{-1} \in \mathrm{sp}(4,\mathbb{R})^{S^1} \,\big|\, P \in \mathrm{Sp}(4,\mathbb{R})^{S^1}\}$, where $X^\alpha = \begin{pmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{pmatrix}$ with $\alpha \neq 0$. Then, $\mathscr{S}$ is the set of all S^1-equivariant real infinitesimally symplectic 4×4 matrices with the same symplectic shape as $X^\lambda = X_{\gamma^+}$. Let $\mathscr{S}_0$ be the open subset of $\mathscr{S}$ where $|\alpha - \lambda|$ is small. Then, using the same argument as in the proof of corollary 3.2, it follows that every smooth curve $\widetilde{\gamma} \mapsto X_{\widetilde{\gamma}}$ in $\mathrm{sp}(4,\mathbb{R})^{S^1}$, which is sufficiently close to $\gamma \mapsto X_\gamma$, intersects $\mathscr{S}_0$ transversely. Thus we have shown the following.

CLAIM 3.5.– For all γ near γ^+, the smooth curve $\gamma \mapsto X_\gamma$ is $\mathrm{sp}(4,\mathbb{R})^{S^1}$-typical.

In other words, the switch twist bifurcation of $\gamma \mapsto X_\gamma$ near γ^+ is S^1-equivariantly stable, that is it cannot be removed by a small S^1-equivariant infinitesimally symplectic perturbation.

Combining claim 3.3 with claim 3.5 we have shown the following.

CLAIM 3.6.– At $p = p_{1,-1}$, the smooth S^1-equivariant curve $\mathscr{X} : [0,1] \to \mathrm{sp}(4,\mathbb{R})^{S^1}$: $\gamma \mapsto X_\gamma = DX_{H_\gamma}(p)$ is $\mathrm{sp}(4,\mathbb{R})^{S^1}$-typical.

3.3.4. *Sign exchange bifurcation*

In this section, we show that the family $\mathscr{X} : [0,1] \to \mathrm{sp}(4,\Omega) : \gamma \mapsto DX_{H_\gamma}(p_{-1,-1}) = X_\gamma$ undergoes an S^1-equivariant sign exchange bifurcation at $\gamma = \gamma^-$.

Consider the S^1-action on $(\mathbb{R}^4, \omega = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right))$ defined by

$$\Phi : S^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^4 \to \mathbb{R}^4 : \left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \begin{pmatrix} \cos tI & \sin tI \\ -\sin tI & \cos tI \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then, $\Phi^*\omega = \omega$ and Φ_t commutes with $Y_0 = \left(\begin{smallmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ -\beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{smallmatrix}\right)$, where $\beta > 0$.

CLAIM 3.7.– Suppose that $\gamma : \mathbb{R} \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \mu \mapsto Y_\mu$ is a smooth family of real S^1-invariant infinitesimally symplectic mappings with Y_0 as given above. For some open interval I containing 0, there is a smooth mapping $P : I \to \mathrm{Sp}(4,\mathbb{R})^{S^1} : \mu \mapsto P_\mu$ with $P_0 = I$, which transforms the curve γ to the smooth S^1-equivariant normal form $\Gamma : I \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \mu \mapsto Z_\mu = P_\mu^{-1} Y_\mu P_\mu$, where

$$Z_\mu = \begin{pmatrix} 0 & 0 & \beta + \nu_1(\mu) & 0 \\ 0 & 0 & 0 & -(\beta - \nu_2(\mu)) \\ -(\beta + \nu_1(\mu)) & 0 & 0 & 0 \\ 0 & \beta - \nu_2(\mu) & 0 & 0 \end{pmatrix}, \qquad [3.26]$$

where for $i = 1, 2$, the function $\nu_i : I \to \mathbb{R} : \mu \mapsto \nu_i(\mu)$ is smooth with $\nu_i(0) = 0$.

LEMMA 3.4.–

$$\mathrm{sp}(4,\mathbb{R})^{S^1} = \left\{ X = \begin{pmatrix} 0 & -a & b & c \\ a & 0 & c & d \\ -b & -c & 0 & -a \\ -c & -d & a & 0 \end{pmatrix} \in \mathrm{sp}(4,\mathbb{R}) \,\middle|\, (a,b,c,d) \in \mathbb{R}^4 \right\}. \qquad [3.27]$$

PROOF.– It is sufficient and straightforward to show that if $XY_0 = Y_0X$ with $X \in \mathrm{sp}(4,\mathbb{R})$, then X has the form given in the right-hand side of equation [3.27].

LEMMA 3.5.– Let Y_0 be as given above. Then

$$\mathrm{sp}(4,\mathbb{R})^{S^1} = \mathrm{im}\,\mathrm{ad}_{Y_0} \oplus \mathrm{span}\{R, T\}, \qquad [3.28]$$

where $R = \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{smallmatrix}\right)$.

PROOF.– A short calculation shows that

$$\mathrm{im}\,\mathrm{ad}_{Y_0} = \left\{ \begin{pmatrix} 0 & -a & 0 & c \\ a & 0 & c & 0 \\ 0 & -c & 0 & -a \\ -c & 0 & a & 0 \end{pmatrix} \in \mathrm{sp}(4,\mathbb{R})^{S^1} \,\middle|\, (a,c) \in \mathbb{R}^2 \right\}.$$

Equation [3.28] follows immediately.

PROOF (of claim 3.7).– Consider the mapping

$$\Psi : \mathbb{R}^2 \times \mathrm{Sp}(4,\mathbb{R})^{S^1} \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \big((\nu_1,\nu_2),P\big) \mapsto P(Y_0+\nu_1 R+\nu_2 T)P^{-1}.$$

Then, using equation [3.28] we see that the map

$$D\Psi(0,I) : \mathbb{R}^2 \times \mathrm{sp}(4,\mathbb{R})^{S^1} \to \mathrm{sp}(4,\mathbb{R})^{S^1} : \big((s,t),X\big) \mapsto \mathrm{ad}_{Y_0}X + sR + tT$$

is surjective. Using the implicit function theorem, the claim follows.

Next we apply claim 3.7 to the smooth mapping $\mathscr{X} : [0,1] \to \mathrm{sp}(4,\Omega) : \gamma \mapsto DX_{H_\gamma}(p_{-1,-1}) = X_\gamma$, where $\gamma \in I$, an open interval containing γ^-.

Since $\widehat{\Delta} > 0$ at $p = p_{-1,-1}$, the characteristic polynomial of X_γ [3.8] may be factored as

$$\big(x^2 + \tfrac{1}{4}(\mu+\alpha+\sqrt{\widehat{\Delta}})^2\big)\big(x^2 + \tfrac{1}{4}(\mu+\alpha-\sqrt{\widehat{\Delta}})^2\big),$$

and the characteristic polynomial of the smooth normal form Z_γ [3.26] is

$$\big(x^2 + (\beta+\nu_1)^2\big)\big(x^2 + (-\beta+\nu_2)^2\big),$$

where $\beta = \sqrt{\alpha(\alpha-\nu)}\Big|_{\gamma=\gamma^-} = \frac{1}{s}\sqrt{(1-2\gamma^-)(1-\gamma^-)}$. Since X_γ and Z_γ are conjugate, their characteristic polynomials are equal, that is

$$\beta+\nu_1 = \tfrac{1}{2}(\mu+\alpha) + \tfrac{1}{2}\sqrt{\widehat{\Delta}} = \beta_+ \text{ and } -\beta+\nu_2 = \tfrac{1}{2}(\mu+\alpha) - \tfrac{1}{2}\sqrt{\widehat{\Delta}} = \beta_-.$$

In other words, $\nu_1(\gamma) = -\beta + \beta_+$ and $\nu_2(\gamma) = \beta + \beta_-$. Since $\beta_\pm = \frac{1}{2}(\mu+\alpha) \pm \sqrt{\frac{1}{4}(\mu+\alpha)^2 - \mu(\alpha-\nu)}$ and $(\mu+\alpha)\Big|_{\gamma=\gamma^-} = 0$, it follows that $\beta_\pm\Big|_{\gamma=\gamma^-} = \pm\sqrt{\alpha(\alpha-\nu)}\ \Big|_{\gamma=\gamma^-} = \pm\beta$. Therefore, $\nu_1(\gamma^-) = \nu_2(\gamma^-) = 0$. So, the S^1-equivariant smooth normal form of X_γ for $\gamma \in I$ is

$$Z_\gamma = X_{\gamma^-} + \nu_1(\gamma)R + \nu_2(\gamma)T = \begin{pmatrix} 0 & 0 & \beta_+ & 0 \\ 0 & 0 & 0 & \beta_- \\ -\beta_+ & 0 & 0 & 0 \\ 0 & -\beta_- & 0 & 0 \end{pmatrix}.$$

Since $\mu+\alpha=\frac{1}{s\gamma^-}(\gamma^- - \gamma)$ implies that $\mu+\alpha>0$, when $\gamma<\gamma^-$ and $\mu+\alpha<0$, when $\gamma>\gamma^-$, it follows that

$$\beta_+ = \tfrac{1}{2}(\mu+\alpha)+\sqrt{\widehat{\Delta}} > -\tfrac{1}{2}(\mu+\alpha)+\sqrt{\widehat{\Delta}} = -\beta_-,$$

when $\gamma<\gamma^-$, and

$$\beta_+ = \tfrac{1}{2}(\mu+\alpha)+\sqrt{\widehat{\Delta}} < -\tfrac{1}{2}(\mu+\alpha)+\sqrt{\widehat{\Delta}} = -\beta_-,$$

when $\gamma>\gamma^-$. Therefore, Z_γ, and hence X_γ, undergoes at least one sign exchange bifurcation as γ increases through γ^-. This sign exchange bifurcation is *unique* because on the open interval I, the function $\gamma\mapsto\beta_+(\gamma)$ is strictly decreasing, while the function $\gamma\mapsto-\beta_-(\gamma)$ is strictly increasing, and both are equal to β when $\gamma=\gamma^-$. This is a consequence of the following.

LEMMA 3.6.– The functions $\gamma\mapsto\beta_\pm$ are strictly decreasing on the open interval I containing γ^- and take the value $\pm\beta$ at $\gamma=\gamma^-$.

PROOF.– Clearly

$$\beta_\pm(\gamma)=\tfrac{1}{2s\gamma^-}(\gamma^- - \gamma)\pm\tfrac{1}{2}\sqrt{\left(\tfrac{1}{s}(1-2\gamma)-\tfrac{1}{\ell}\gamma\right)^2+\tfrac{4}{s\ell}\gamma^2}$$

and $\beta_\pm(\gamma^-)=\pm\beta$. Since $s=\varepsilon^2\ell$, we obtain

$$\begin{aligned}\widehat{\Delta} &= \tfrac{1}{s^2\ell^2}\left((\ell(1-2\gamma)-s\gamma)^2+4s\ell\gamma^2\right)=\tfrac{1}{\ell^2\varepsilon^4}\left(((1-2\gamma)-\varepsilon^2\gamma)^2+4\varepsilon^2\gamma^2\right)\\ &= \tfrac{1}{\ell^2\varepsilon^4}\left((1-2\gamma)^2+\mathrm{O}(\varepsilon^2)\right)\end{aligned}$$

and therefore $\sqrt{\widehat{\Delta}}=\frac{1}{\ell\varepsilon^2}\left((1-2\gamma)+\mathrm{O}(\varepsilon^2)\right)$. Using $\frac{\mathrm{d}\widehat{\Delta}}{\mathrm{d}\gamma}=\frac{1}{\ell^2\varepsilon^4}\left(-4(1-2\gamma)+\mathrm{O}(\varepsilon^2)\right)$ and $\frac{\mathrm{d}(\mu+\alpha)}{\mathrm{d}\gamma}=-\frac{1}{s\gamma^-}$, we find that

$$\frac{\mathrm{d}\beta_\pm}{\mathrm{d}\gamma}=\tfrac{1}{2}\frac{\mathrm{d}(\mu+\alpha)}{\mathrm{d}\gamma}\pm\tfrac{1}{4}\frac{1}{\sqrt{\widehat{\Delta}}}\frac{\mathrm{d}\widehat{\Delta}}{\mathrm{d}\gamma}=-\tfrac{1}{2\ell\gamma^-\varepsilon^2}(1\pm2\gamma^-)+\mathrm{O}(1).$$

Since $(1\pm2\gamma^-)>0$, we have $\frac{\mathrm{d}\beta_\pm}{\mathrm{d}\gamma}<0$ for all $\gamma\in I$. Thus, as γ increases through γ^-, the functions $\gamma\mapsto\beta_\pm(\gamma)$ strictly decrease through β.

From the proof of lemma 3.6, it follows that $\frac{\mathrm{d}\nu_1}{\mathrm{d}\gamma}=\frac{\mathrm{d}\beta_+}{\mathrm{d}\gamma}<0$. Thus, using the argument in the proof of corollary 3.1, the smooth curve $\gamma\mapsto X_\gamma=DX_{H_\gamma}(p_{-1,-1})$

crosses the $\mathrm{Sp}(4,\mathbb{R})^{S^1}$-adjoint orbit $\mathscr{O}_{X_{\gamma^-}}$ transversely. This result can be improved. Let $\mathscr{S} = \{PX^{\alpha}P^{-1} \in \mathrm{sp}(4,\mathbb{R})^{S^1} \mid P \in \mathrm{Sp}(4,\mathbb{R})^{S^1}\}$, where $X^{\alpha} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\alpha \\ -\alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}$ with $\alpha \neq 0$. Then, $\mathscr{S}$ is the set of all S^1-equivariant real infinitesimally symplectic 4×4 matrices with the same symplectic shape as $X^{\beta} = X_{\gamma^-}$. Consider the subset $\mathscr{S}_0$ of $\mathscr{S}$ where $|\alpha - \beta|$ is small. Then applying the argument used to prove corollary 3.2 it follows that every smooth curve $\widetilde{\gamma} \mapsto X_{\widetilde{\gamma}}$ in $\mathrm{sp}(4,\mathbb{R})^{S^1}$, which is sufficiently close to $\gamma \mapsto X_{\gamma}$, intersects $\mathscr{S}_0$ transversely. In other words, we have shown the following.

CLAIM 3.8.– For all γ near γ^-, the smooth curve $\gamma \mapsto X_{\gamma}$ is $\mathrm{sp}(4,\mathbb{R})^{S^1}$-typical.

In other words, the sign exchange bifurcation of $\mathscr{X};\gamma \mapsto X_{\gamma}$ near γ^- is S^1-equivariantly stable, that is it can not be removed by a small S^1-equivariant infinitesimally symplectic perturbation.

In claim 5.2 below, we show that when $p = p_{1,-1}$, the mapping $\mathscr{X}$ undergoes a global S^1-equivariant sign exchange bifurcation when γ passes from $[0,\gamma_-)$ to $(\gamma_+,1]$.

3.4. Infinitesimally symplectic normal form

In this section, we compute the infinitesimally symplectic normal form of X_{γ} [3.5] at the bifurcation values $\gamma^{\pm}$ and $\gamma_{\pm}$ of γ.

3.4.1. *Normal form of* X_{γ} *at* $\gamma^{\pm}$

To determine the normal form at $\gamma^{\pm}$, we first find a complexification of X_{γ} and then determine its eigenvalues. We start by identifying $\mathbb{R}^4$ with coordinates (x_1,x_2,y_1,y_2) with $\mathbb{C}^2$ having coordinates $(z = x_1 + ix_2, w = y_1 + iy_2)$. Then, the real 4×4 matrix X_{γ} becomes the 2×2 complex matrix $x_{\gamma} = \begin{pmatrix} i\alpha & -i\mu \\ -i\nu & i\mu \end{pmatrix}$. The characteristic polynomial of x_{γ} is

$$x^2 - i(\mu+\alpha)x + \mu(\nu-\alpha),$$

whose roots are $\lambda_{\pm} = \frac{1}{2}\,i(\mu+\alpha) \pm \frac{1}{2}\sqrt{\widehat{\Delta}}$. The matrix x_{γ} has multiple eigenvalues

$$\begin{cases} \frac{1}{2}\,i(\mu+\alpha), & \text{when } \widehat{\Delta} = 0, \text{ that is when } \gamma = \gamma_{\pm} \\ \pm\sqrt{-\alpha(\alpha-\nu)}, & \text{when } \mu+\alpha = 0, \text{ that is when } \gamma = \gamma^{+} \\ \pm i\sqrt{\alpha(\alpha-\nu)}, & \text{when } \mu+\alpha = 0, \text{ that is when } \gamma = \gamma^{-}. \end{cases}$$

The complex eigenvector $\binom{z_\pm}{w_\pm}$ of x_γ corresponding to the eigenvalue $\lambda_\pm$ satisfies

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix} i\alpha-\lambda_\pm & -i\mu \\ -i\nu & i\mu-\lambda_\pm \end{pmatrix}\begin{pmatrix}z_\pm\\w_\pm\end{pmatrix} = \begin{pmatrix}(i\alpha-\lambda_\pm)z_\pm - i\mu w_\pm \\ -i\nu z_\pm + (i\mu-\lambda_\pm)w_\pm\end{pmatrix}.$$

Only one of the above equations matters, namely $i\mu w_\pm = (i\alpha-\lambda_\pm)z_\pm$. Thus, the complex eigenspace $E_{\lambda_\pm}$ of x_γ corresponding to the eigenvalue $\lambda_\pm$ is $\mathrm{span}_{\mathbb{C}}\binom{\mu}{\alpha+i\lambda_\pm} = \left\{(v_1+iv_2)\binom{\mu}{\alpha-\mathrm{Im}\,\lambda_\pm+i\mathrm{Re}\,\lambda_\pm}\Big|\,(v_1,v_2)\in\mathbb{R}^2\right\}$. Since

$$E_{\lambda_\pm} = \left\{\begin{pmatrix}\mu v_1 + i\mu v_2 \\ \big((\alpha-\mathrm{Im}\,\lambda_\pm)v_1 - \mathrm{Re}\,\lambda_\pm v_2\big) + i\big(\mathrm{Re}\,\lambda_\pm v_1 + (\alpha-\mathrm{Im}\,\lambda_\pm)v_2\big)\end{pmatrix} \in \mathbb{C}^2 \,\Big|\, (v_1,v_2)\in\mathbb{R}^2\right\},$$

the real X_γ-invariant two-plane associated with the complex eigenspace $E_{\lambda_\pm}$ is

$$\Pi_\gamma^\pm = \mathrm{span}_{\mathbb{R}}\left\{e_\gamma^\pm = \begin{pmatrix}\mu\\0\\\alpha-\mathrm{Im}\,\lambda_\pm\\\mathrm{Re}\,\lambda_\pm\end{pmatrix}, f_\gamma^\pm = \begin{pmatrix}0\\\mu\\-\mathrm{Re}\,\lambda_\pm\\\alpha-\mathrm{Im}\,\lambda_\pm\end{pmatrix}\right\}.$$

The following calculation checks that $\Pi_\gamma^\pm$ is indeed X_γ-invariant. We have

$$\big(X_\gamma - (\mathrm{Re}\,\lambda_\pm)I\big)e_\gamma^\pm = \begin{pmatrix}0\\ \mu\,\mathrm{Im}\,\lambda_\pm \\ -(\alpha+\mu)\mathrm{Re}\,\lambda_\pm + \mathrm{Re}\,\lambda_\pm\,\mathrm{Im}\,\lambda_\pm \\ \mu(\alpha-\nu) - \mu\,\mathrm{Im}\,\lambda_\pm - (\mathrm{Re}\,\lambda_\pm)^2\end{pmatrix} = (\mathrm{Im}\,\lambda_\pm)\,f_\gamma^\pm. \qquad [3.29]$$

The last equality above follows from the identities

$$\mathrm{Re}\,\lambda_\pm\big(2\,\mathrm{Im}\,\lambda_\pm - (\mu+\alpha)\big) = 0 \qquad [3.30]$$

$$(\mathrm{Re}\,\lambda_\pm)^2 - (\mathrm{Im}\,\lambda_\pm)^2 + (\mu+\alpha)\,\mathrm{Im}\,\lambda_\pm + \mu(\nu-\alpha) = 0. \qquad [3.31]$$

These equalities hold because they are the imaginary and real parts of the equation $\lambda_\pm^2 - i(\mu+\alpha)\lambda_\pm + \mu(\nu-\alpha) = 0$. Similarly, we obtain

$$\big(X_\gamma - (\mathrm{Re}\,\lambda_\pm)I\big)f_\gamma^\pm = -(\mathrm{Im}\,\lambda_\pm)e_\gamma^\pm. \qquad [3.32]$$

We now compute the infinitesimally symplectic normal form of X_γ, when $\gamma=\gamma^\pm$, that is when $\mu+\alpha=0$. Using expression [3.5] for X_γ, we find that

$$X_\gamma^2 = \begin{pmatrix} -(\alpha^2+\mu\nu) & 0 & \mu(\alpha+\mu) & 0 \\ 0 & -(\alpha^2+\mu\nu) & 0 & \mu(\alpha+\mu) \\ \nu(\alpha+\mu) & 0 & -(\mu^2+\mu\nu) & 0 \\ 0 & \nu(\alpha+\mu) & 0 & -(\mu^2+\mu\nu)\end{pmatrix}.$$

Since $\mu + \alpha = 0$, we obtain $X_{\gamma^\pm}^2 + \alpha(\alpha - \nu)I = 0$. Consequently, $X_{\gamma^\pm}$ is semi-simple with real eigenvalues

$$\lambda_\pm = \pm\lambda = \pm\sqrt{-\alpha(\alpha-\nu)} = \pm\tfrac{1}{s}\sqrt{(2\gamma^+ - 1)(1-\gamma^+)}, \qquad [3.33]$$

when $\gamma = \gamma^+$ and purely imaginary eigenvalues

$$\lambda_\pm = \pm i\beta = \pm i\sqrt{\alpha(\alpha-\nu)} = \pm\tfrac{1}{s}i\sqrt{(1-2\gamma^-)(1-\gamma^-)}, \qquad [3.34]$$

when $\gamma = \gamma^-$.

From [3.29] and [3.32], we see that $\{e_{\gamma^+}^\pm, f_{\gamma^+}^\pm\} = \left\{ \begin{pmatrix} \mu \\ 0 \\ \alpha \\ \pm\lambda \end{pmatrix}, \begin{pmatrix} 0 \\ \mu \\ \mp\lambda \\ \alpha \end{pmatrix} \right\}$ is a basis for the eigenspace of X_{γ^+} corresponding to the real eigenvalue $\pm\lambda$, while $\{e_{\gamma^-}^\pm, f_{\gamma^-}^\pm\} = \left\{ \begin{pmatrix} \mu \\ 0 \\ \alpha\mp\beta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu \\ 0 \\ \alpha\mp\beta \end{pmatrix} \right\}$ is a basis for the real eigenplane $\Pi_{\gamma^-}^\pm$ of X_{γ^-} with respect to which the matrix of $X_{\gamma^-}|\Pi_{\gamma^-}^\pm$ is $\begin{pmatrix} 0 & \mp\beta \\ \pm\beta & 0 \end{pmatrix}$.

We now look at the symplectic structure of the eigenspaces of X_{γ^+}. We have $\Omega(e_{\gamma^+}^+, e_{\gamma^+}^-) = \Omega(f_{\gamma^+}^+, f_{\gamma^+}^-) = 0$,

$$\begin{aligned}\Omega(e_{\gamma^+}^\pm, f_{\gamma^+}^\pm) &= \mu^2\Omega(e_1,e_2) + (\alpha^2+\lambda^2)\Omega(e_3,e_4) \\ &= -\tfrac{1}{s}\mu^2 + \tfrac{1}{\ell}(\alpha^2 - \alpha^2 + \alpha\nu), \quad \text{using [3.6] and } \eta_s = 1 = -\eta_\ell \\ &= -\mu(\tfrac{1}{s}\mu + \tfrac{1}{\ell}\nu) = 0, \quad \text{since } \mu = \tfrac{1}{\ell}\gamma^+ = -\alpha \text{ and } \nu = -\tfrac{1}{s}\gamma^+,\end{aligned}$$

and

$$\begin{aligned}\Omega(e_{\gamma^+}^\pm, f_{\gamma^+}^\mp) &= -\tfrac{1}{s}\mu^2 + \tfrac{1}{\ell}(\alpha^2+\alpha^2-\alpha\nu) = -\mu\left(\tfrac{\mu}{s} - \tfrac{\nu}{\ell}\right) + \tfrac{2\alpha^2}{\ell} \\ &= -\tfrac{2(\gamma^+)^2}{\varepsilon^2\ell^3} + \tfrac{2(1-2\gamma^+)^2}{\varepsilon^4\ell^3} = \sigma > 0, \quad \text{since } s = \varepsilon^2\ell \text{ and } 0 < \varepsilon \ll 1.\end{aligned}$$

Therefore, with respect to the basis $\{\frac{1}{\sqrt{\sigma}}e_{\gamma^+}^+, \frac{1}{\sqrt{\sigma}}f_{\gamma^+}^+, \frac{1}{\sqrt{\sigma}}f_{\gamma^+}^-, -\frac{1}{\sqrt{\sigma}}e_{\gamma^+}^-\}$, the matrix of the symplectic form Ω is $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, while the matrix of X_{γ^+} is $\begin{pmatrix} \lambda I & 0 \\ 0 & -\lambda I \end{pmatrix}$, where $\lambda > 0$ and is given by [3.33]. This verifies first entry in Table 3.1.

Next we look at the symplectic structure of the eigenplanes of X_{γ^-}. We have $\Omega(e_{\gamma^-}^+, e_{\gamma^-}^-) = \Omega(f_{\gamma^-}^+, f_{\gamma^-}^-) = 0$,

$$\Omega(e_{\gamma^-}^\pm, f_{\gamma^-}^\mp) = \mu^2\Omega(e_1,e_2) + (\alpha^2-\beta^2)\Omega(e_3,e_4)$$

$$= \tfrac{1}{s}\mu^2 + \tfrac{1}{\ell}(\alpha^2 - \alpha^2 + \alpha\nu), \quad \text{since } \eta_s = -1 = \eta_\ell$$
$$= \mu\left(\tfrac{\mu}{s} - \tfrac{\nu}{\ell}\right) = 0, \quad \text{since } \mu = -\tfrac{\gamma^-}{\ell} = -\alpha \text{ and } \nu = -\tfrac{\gamma^-}{s}$$

and

$$\Omega(e^{\pm}_{\gamma^+}, f^{\pm}_{\gamma^+}) = \tfrac{1}{s}\mu^2 + \tfrac{1}{\ell}(\alpha \mp \beta)^2 = \tau_\pm > 0.$$

Therefore, with respect to the basis $\{\frac{1}{\sqrt{\tau_+}}e^+_{\gamma^+}, \frac{1}{\sqrt{\tau_-}}e^-_{\gamma^+}, \frac{1}{\sqrt{\tau_+}}f^+_{\gamma^+}, \frac{1}{\sqrt{\tau_-}}f^-_{\gamma^+}\}$, the matrix of the symplectic form Ω is $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, while the matrix of X_{γ^-} is $\begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\beta \\ -\beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}$, where $\beta > 0$ and is given by [3.34]. This verifies third entry in Table 3.1.

3.4.2. *Normal form of* X_γ *at* $\gamma_\pm$

Here we compute the infinitesimally symplectic normal form of X_γ [3.5] when $\gamma = \gamma_\pm$, that is when $\widehat{\Delta} = 0$.

We start by proving the following.

LEMMA 4.1.– Suppose that A is a linear mapping of a finite-dimensional real vector space into itself such that $(A^2 + \beta^2 I)^2 = 0$ with $\beta > 0$, but $A^2 + \beta^2 I \neq 0$. Then, $S = A(I + \frac{1}{2\beta^2}(A^2 + \beta^2 I))$ and $N = -\frac{1}{2\beta^2}A(A^2 + \beta^2 I)$ is the semi-simple-nilpotent decomposition of the linear map A, which is unique.

PROOF.– It is straightforward to verify that $S^2 + \beta^2 I = 0$. Thus, S is the semi-simple part of A and $N = A - S$ is the nilpotent part. The proof of uniqueness is omitted.

If $A \in \text{sp}(4, \Omega)$, then N and S lie in $\text{sp}(4, \Omega)$.

When $\gamma = \gamma_\pm$, the eigenvalues of $X_{\gamma_\pm}$ are $\pm i\beta_\pm$ each of multiplicity 2, where

$$\beta_\pm = \tfrac{1}{2}(\mu + \alpha) = \tfrac{1}{2}\left(\tfrac{1}{\ell}\gamma_\pm + \tfrac{1}{s}(1 - 2\gamma_\pm)\right),$$
$$\text{since } \eta_s = 1 \text{ and } \eta_\ell = -1,\ \alpha = \tfrac{1}{s}(1 - 2\gamma_\pm),\ \mu = \tfrac{1}{\ell}\gamma_\pm \text{ and } \nu = -\tfrac{1}{s}\gamma_\pm$$
$$= \mp\tfrac{1}{2\varepsilon\ell}\left(\tfrac{1 \mp \varepsilon}{1 \mp \varepsilon + \frac{1}{2}\varepsilon^2}\right), \text{ since } \gamma_\pm = \tfrac{\ell}{s + 2\ell \mp 2\sqrt{s\ell}} \text{ and } s = \varepsilon^2 \ell,$$
$$= \mp\tfrac{1}{2\varepsilon\ell}(1 \mp \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)). \qquad [3.35]$$

Now

$$N = -\tfrac{1}{2\beta_\pm^2}X_{\gamma_\pm}(X^2_{\gamma_\pm} + \beta_\pm^2 I) = -\tfrac{1}{2\beta_\pm^2}\begin{pmatrix} 0 & -a & 0 & b \\ a & 0 & b & 0 \\ 0 & c & 0 & -a \\ c & 0 & a & 0 \end{pmatrix}, \qquad [3.36]$$

where $a = \alpha(\beta_\pm^2 - \alpha^2) - 2\mu\nu\alpha - \mu^2\nu$, and b and c are polynomials in α, μ and ν. Here, N is the nilpotent part of $X_{\gamma_\pm}$.

We suppose that $A \in \mathrm{sp}(4,\Omega)$ and $A = S + N$, where $SN = NS$, $S^2 + \beta^2 I = 0$ with $\beta > 0$, and $N^2 = 0$. Then, $S, N \in \mathrm{sp}(4,\Omega)$, that is for every $v, w \in \mathbb{R}^4$

$$\Omega(Sv,w) = -\Omega(v,Sw) \quad \text{and} \quad \Omega(Nv,w) = -\Omega(v,Nw).$$

LEMMA 4.2.– Moreover, suppose that there is a non-zero vector $e \in \mathbb{R}^4$ such that $\Omega(e, \frac{1}{\rho}Ne) = 1$ with $\rho^2 = 1$ and $\Omega(e, \frac{1}{\beta}Se) = 0$. With respect to the basis $\mathfrak{e} = \{e, \frac{1}{\beta}Se, \frac{1}{\rho}Ne, \frac{1}{\rho\beta}NSe\}$, the matrix of Ω is $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and the matrix A is $\begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \rho & 0 & 0 & -\beta \\ 0 & \rho & \beta & 0 \end{pmatrix}$, which is in infinitesimally symplectic normal form.

PROOF.– As the verification of the matrix of A with respect to the basis $\mathfrak{e}$ is straightforward, we need only to show that the matrix of Ω with respect to $\mathfrak{e}$ is as desired. This follows from the skew symmetry of Ω, the fact that S and N are infinitesimally symplectic, $S^2 + \beta^2 I = 0$, and $N^2 = 0$.

The following claim together with lemma 4.1 gives a practical construction for the infinitesimally symplectic normal form of A.

CLAIM 4.1.– Suppose that there is a non-zero vector $e \in \mathbb{R}^4$ such that $\Omega(e,Ne) = \rho$, where $\rho^2 = 1$. Let $f = e + \frac{1}{2}\rho\,\Omega(e, \frac{1}{\beta}Se)\,\frac{1}{\beta}SNe$. Then, $\Omega(f,Nf) = \rho$ and $\Omega(f, \frac{1}{\beta}Sf) = 0$.

PROOF.– This is a straightforward calculation.

We now calculate the infinitesimally symplectic normal form of $X_{\gamma_\pm}$, where $\gamma_\pm = \frac{1}{2 \mp 2\varepsilon + \varepsilon^2}$. Since $S^2 + \beta_\pm^2 I = 0$, where S is the semi-simple part of $X_{\gamma_\pm}$ with $\beta_\pm = \frac{1}{2}(\mu + \alpha)$ [3.35], we need only to calculate $\Omega(e_1, Ne_1)$, where N [3.36] is the nilpotent part of $X_{\gamma_\pm}$. Now by defintion of N, we get

$$\Omega(e_1,Ne_1) = \Omega(e_1, \tfrac{1}{2\beta_\pm^2}(-ae_2 - ce_4)) = -\tfrac{a}{2\beta_\pm^2}\Omega(e_1,e_2) = \frac{a}{2\beta_\pm^2 s} = \frac{a}{2\beta_\pm^2 \ell\varepsilon^2}.$$

The second to last equation above follows because $\Omega(e_1,e_4) = 0$ and $\Omega(e_1,e_2) = -\frac{1}{s}$. So at $\gamma = \gamma_\pm$, we have

$$\begin{aligned} a &= \alpha(\beta_\pm^2 - \alpha^2) - 2\mu\alpha\nu - \mu^2\nu = \tfrac{1}{4}(\mu+\alpha)^2(\alpha - 4\nu) + \alpha^2(\nu - \alpha) \\ &= \tfrac{1}{4}\big(\tfrac{1}{s} + \tfrac{s-2\ell}{s\ell}\gamma_\pm\big)^2\big(\tfrac{1}{s}(1-2\gamma_\pm) + \tfrac{4}{s}\gamma_\pm\big) - \tfrac{1}{s^3}(1-2\gamma_\pm)^2(1-\gamma_\pm) \end{aligned}$$

$$\text{using } \alpha = \tfrac{1}{s}(1-2\gamma_\pm),\ \mu = \tfrac{1}{\ell}\gamma_\pm \text{ and } \nu = -\tfrac{1}{s}\gamma_\pm$$
$$= \tfrac{1}{s^3}\Big[(1-2\gamma_\pm)\big(\tfrac{1}{2}\,\tfrac{s}{\ell}\gamma_\pm(1+2\gamma_\pm) - \tfrac{3}{4}(1-2\gamma_\pm)^2\big) + \tfrac{1}{4}\,\tfrac{s^2}{\ell^2}\gamma_\pm^2(1+2\gamma_\pm)\Big].$$

Since $\gamma_\pm = \frac{1}{2}\,\frac{1}{1\mp\varepsilon+\frac{1}{2}\varepsilon^2}$, we get $1-2\gamma_\pm = \mp\varepsilon + (1\mp\frac{1}{2})\varepsilon^2 + \mathrm{O}(\varepsilon^3)$. So

$$a = \frac{1}{\ell^3\varepsilon^6}\left[\mp\varepsilon(1+\mathrm{O}(\varepsilon))\big(\tfrac{1}{2}\,\gamma_\pm(1+2\gamma_\pm)\varepsilon^2 - \tfrac{3}{4}\varepsilon^2(1+\mathrm{O}(\varepsilon))\big) + \right.$$
$$\left. + \tfrac{1}{4}\gamma_\pm^2(1+2\gamma_\pm)\varepsilon^4\right] = \mp\frac{1}{\ell^3\varepsilon^3}\big((\tfrac{1}{2}\,\gamma_\pm(1+2\gamma_\pm) - \tfrac{3}{4}) + \mathrm{O}(\varepsilon)\big).$$

Since $\frac{1}{2}\gamma_\pm(1+2\gamma_\pm) - \frac{3}{4} = -\frac{1}{4} + \mathrm{O}(\varepsilon)$, we obtain $a = \frac{\pm 1}{4\ell^3\varepsilon^3}(1+\mathrm{O}(\varepsilon))$. Consequently, when $0 < \varepsilon \ll 1$

$$\rho_\pm = \operatorname{sgn}\Omega(e_1, Ne_1) = \operatorname{sgn}\left[\frac{1}{2\beta_\pm^2\ell\varepsilon^2}\cdot\frac{\pm 1}{4\ell^3\varepsilon^3}\right] = \begin{cases} 1, & \text{at } \gamma = \gamma_+ \\ -1, & \text{at } \gamma = \gamma_-. \end{cases} \qquad [3.37]$$

Thus, the infinitesimally symplectic normal form of $X_{\gamma_\pm}$ at $p_{1,-1}$ is $\begin{pmatrix} 0 & \mp\beta_\pm & 0 & 0 \\ \pm\beta_\pm & 0 & 0 & 0 \\ \rho_\pm & 0 & 0 & \mp\beta_\pm \\ 0 & \rho_\pm & \pm\beta_\pm & 0 \end{pmatrix}$ with $\mp\beta_\pm = \frac{1}{2\varepsilon\ell}\left(\frac{1\mp\varepsilon}{1\mp\varepsilon+\frac{1}{2}\varepsilon^2}\right) > 0$ and $\rho_\pm = \begin{cases} 1, \text{ when } \gamma=\gamma_+ \\ -1, \text{ when } \gamma=\gamma_-, \end{cases}$ where $\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Both of the preceding matrices are taken with respect to the basis $\{f,\, \frac{\pm 1}{\beta_\pm}Sf,\, \frac{1}{\rho_\pm}Nf,\, \frac{\pm 1}{\rho_\pm\beta_\pm}SNf\}$. Here, $e = \sqrt{\frac{2\ell}{|A|}}\beta_\pm\varepsilon\, e_1$ is defined so that $\Omega(e, Ne) = \rho_\pm$ and $f = e + \frac{1}{2}\rho_\pm\Omega(e, \frac{1}{\beta_\pm}Se)\frac{1}{\beta_\pm}SNe$. This verifies second entry in Table 3.1 and thus the proof of claim 3.1.

3.5. Global issues

In this section, we discuss some global issues concerning the smooth family $\mathscr{X} : [0,1] \to \mathrm{sp}(4,\Omega) : \gamma \mapsto DX_{H_\gamma}(p) = X_\gamma$, where $p = p_{\pm 1,-1}$.

3.5.1. *Invariant Lagrange planes*

In this section, we discuss global issues involving X_γ-invariant Lagrange planes. We prove the following.

LEMMA 5.1.– Let $\Lambda_\gamma^\pm = \operatorname{span}\left\{ e_\gamma^\pm = \begin{pmatrix} \mu \\ 0 \\ \alpha - \operatorname{Im}\lambda_\pm \\ \operatorname{Re}\lambda_\pm \end{pmatrix},\, f_\gamma^\pm = \begin{pmatrix} 0 \\ \mu \\ -\operatorname{Re}\lambda_\pm \\ \alpha - \operatorname{Im}\lambda_\pm \end{pmatrix} \right\}$, where $\lambda_\pm = \frac{1}{2}i(\mu + \alpha) \pm \frac{1}{2}\sqrt{\widehat{\Delta}}$ with $\widehat{\Delta} \geq 0$. Then, $\Lambda_\gamma^\pm$ is an X_γ-invariant Lagrange plane in $(\mathbb{R}^4, \Omega)$ for every $\gamma \in [\gamma_-, \gamma_+]$ when $p = p_{1,-1}$.

PROOF.– We compute

$$\Omega_{\pm} = \Omega(e^{\pm}_{\gamma}, f^{\pm}_{\gamma}) = -\tfrac{1}{s}\mu^2 + \tfrac{1}{\ell}\big((\alpha - \operatorname{Im}\lambda_{\pm})^2 + (\operatorname{Re}\lambda_{\pm})^2\big). \qquad [3.38]$$

Suppose that $\operatorname{Re}\lambda_{\pm} \neq 0$, that is $\gamma \in (\gamma_-, \gamma_+)$. From equation [3.30] it follows that $\operatorname{Im}\lambda_{\pm} = \frac{1}{2}(\mu+\alpha)$. Using equation [3.31], we obtain $(\operatorname{Re}\lambda_{\pm})^2 = -\frac{1}{4}(\mu+\alpha)^2 - \mu(\nu - \alpha)$. Therefore,

$$(\alpha - \operatorname{Im}\lambda_{\pm})^2 + (\operatorname{Re}\lambda_{\pm})^2 = -\mu\nu.$$

So $\Omega_{\pm} = -\frac{1}{s}\mu^2 - \frac{1}{\ell}\mu\nu = 0$, since $\mu = \frac{1}{\ell}\gamma$ and $\nu = -\frac{1}{s}\gamma$ at $p_{1,-1}$. From the skew symmetry of Ω, we obtain $\Omega(e^{\pm}_{\gamma}, e^{\pm}_{\gamma}) = \Omega(f^{\pm}_{\gamma}, f^{\pm}_{\gamma}) = 0$. Therefore, $\Lambda^{\pm}_{\gamma}$ is a Lagrange plane, which is invariant under X_{γ} using equations [3.29] and [3.32]. Note that Λ^{+}_{γ} and Λ^{-}_{γ} are transverse.

Now suppose that $\operatorname{Re}\lambda_{\pm} = 0$. Then, $\operatorname{Im}\lambda_{\pm} = \frac{1}{2}(\mu+\alpha)$. If $\widehat{\Delta} = 0$, that is $\gamma = \gamma_{\pm}$, then $(\alpha - \operatorname{Im}\lambda_{\pm})^2 = \frac{1}{4}(\mu-\alpha)^2 = -\mu\nu$. Therefore, $\Omega_{\pm} = 0$. So, $\Lambda^{\pm}_{\gamma_{\pm}}$ is an $X_{\gamma_{\pm}}$-invariant Lagrange plane. Since $\widehat{\Delta} = 0$ and $\operatorname{Re}\lambda_{\pm} = 0$ imply that $\lambda_+ = \lambda_-$ when $\gamma = \gamma_{\pm}$, it follows that $\Lambda^{+}_{\gamma_{\pm}} = \Lambda^{-}_{\gamma_{\pm}}$.

For $\gamma \in (\gamma_-, \gamma_+)$, the eigenvalue

$$\lambda_{\pm} = \tfrac{1}{2}\, i\big(\tfrac{1}{\ell}\gamma + \tfrac{1}{s}(1-2\gamma)\big) \pm \tfrac{1}{2}\sqrt{(\gamma - \gamma_-)(\gamma^+ - \gamma)}$$

is a smooth function, which is continuous at $\gamma = \gamma_{\pm}$. Consequently, $\gamma \mapsto \Lambda^{+}_{\gamma}$ is a smooth function on (γ_-, γ_+), which is continuous at $\gamma = \gamma_{\pm}$. Now the matrix of $X_{\gamma}|\Lambda^{+}_{\gamma}$ with respect to the basis $\{e^{+}_{\gamma}, f^{+}_{\gamma}\}$ is

$$\begin{pmatrix} \operatorname{Re}\lambda_+ & -\operatorname{Im}\lambda_+ \\ \operatorname{Im}\lambda_+ & \operatorname{Re}\lambda_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{-\widehat{\Delta}} & -\frac{1}{2}(\mu+\alpha) \\ \frac{1}{2}(\mu+\alpha) & \frac{1}{2}\sqrt{-\widehat{\Delta}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{(\gamma-\gamma_-)(\gamma_+-\gamma)} & -\frac{1}{2}\big(\frac{1}{\ell}\gamma + \frac{1}{s}(1-2\gamma)\big) \\ \frac{1}{2}\big(\frac{1}{\ell}\gamma + \frac{1}{s}(1-2\gamma)\big) & \frac{1}{2}\sqrt{(\gamma-\gamma_-)(\gamma_+-\gamma)} \end{pmatrix},$$

which is a continuous function of γ on $[\gamma_-, \gamma_+]$. Thus, the rotational part $\begin{pmatrix} 0 & -\frac{1}{2}(\mu+\alpha) \\ \frac{1}{2}(\mu+\alpha) & 0 \end{pmatrix}$ of $X_{\gamma}|\Lambda^{+}_{\gamma}$ is a continuous function on $[\gamma_-, \gamma_+]$. Because $\frac{1}{2}(\mu+\alpha) = \frac{1}{2s\gamma^+}(\gamma^+ - \gamma)$, we have proved the following.

CLAIM 5.1.– X_{γ} at $p = p_{1,-1}$ has a positive twist when $\gamma \in [\gamma_-, \gamma^+)$ and a negative twist when $\gamma \in (\gamma^+, \gamma_+]$.

3.5.2. *Symplectic signs*

In this section, we treat global issues involving X_γ-invariant symplectic planes in $(\mathbb{R}^4, \Omega)$.

LEMMA 5.2.– Let $\Pi_\gamma^\pm = \left\{ e_\gamma^\pm = \begin{pmatrix} \mu \\ 0 \\ \alpha - \operatorname{Im}\lambda_\pm \\ 0 \end{pmatrix}, f_\gamma^\pm = \begin{pmatrix} 0 \\ \mu \\ 0 \\ \alpha - \operatorname{Im}\lambda_\pm \end{pmatrix} \right.$, where $\lambda_\pm = \pm i\beta = i\left(\frac{1}{2}(\mu+\alpha) \pm \frac{1}{2}\sqrt{\widehat{\Delta}}\right)$ with $\widehat{\Delta} > 0$. Then, $\Pi_\gamma^\pm$ is an X_γ-invariant symplectic plane for every $\gamma \in \begin{cases} [0,\gamma_-] \cup [\gamma_+, 1], & \text{when } p = p_{1,-1} \\ (0,1), & \text{when } p = p_{-1,-1}. \end{cases}$

PROOF.– We compute $\Omega_\pm = \Omega(e_\gamma^\pm, f_\gamma^\pm) = -\kappa \frac{1}{s}\mu^2 + \frac{1}{\ell}(\operatorname{Im}\lambda_\pm - \alpha)^2$, where κ is the $\pm$ sign in $p_{\pm,-1}$. When $\kappa = -$, then obviously $\Omega_\pm > 0$. When $\kappa = +$, then

$$\begin{aligned} \Omega_\pm &= -\tfrac{1}{s}\mu^2 + \tfrac{1}{4\ell}(\mu - \alpha \pm \sqrt{\widehat{\Delta}})^2 = -\tfrac{1}{s}\mu^2 + \tfrac{1}{\ell}\mu\nu + \tfrac{1}{2\ell}(\mu-\alpha)(\mu - \alpha \pm \sqrt{\widehat{\Delta}}), \\ &\qquad \text{since } \widehat{\Delta} = (\mu-\alpha)^2 + 4\mu\nu \\ &= \tfrac{1}{2\ell}(\mu-\alpha)(\mu-\alpha \pm \sqrt{\widehat{\Delta}}), \quad \text{since } \mu = \tfrac{1}{\ell}\gamma \text{ and } \nu = -\tfrac{1}{s}\gamma \text{ at } p_{1,-1}. \end{aligned}$$

Because $\mu\nu < 0$, we get $0 < \widehat{\Delta} = (\mu-\alpha)^2 + 4\mu\nu < (\mu-\alpha)^2$, which implies $|\mu - \alpha| > \sqrt{\widehat{\Delta}}$. Therefore, $\begin{cases} \mu - \alpha \pm \sqrt{\widehat{\Delta}} > 0, & \text{if } \mu - \alpha > \sqrt{\widehat{\Delta}} > 0 \\ \mu - \alpha \pm \sqrt{\widehat{\Delta}} < 0, & \text{if } \mu - \alpha < -\sqrt{\widehat{\Delta}} < 0. \end{cases}$ Consequently, $\Omega_\pm > 0$. Because Ω is skew symmetric, it follows that $\Omega(e_\gamma^\pm, e_\gamma^\pm) = \Omega(f_\gamma^\pm, f_\gamma^\pm) = 0$. Thus, with respect to the basis $\{\frac{1}{\sqrt{\Omega_\pm}} e_\gamma^\pm, \frac{1}{\sqrt{\Omega_\pm}} f_\gamma^\pm\}$, the matrix of $\Omega|\Pi_\gamma^\pm$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Moreover, using equations [3.29] and [3.32], we find that the matrix of $X_\gamma|\Pi_\gamma^\pm$ is $\begin{pmatrix} 0 & -\beta_\pm \\ \beta_\pm & 0 \end{pmatrix}$, where

$$\begin{aligned} \beta_\pm &= \tfrac{1}{2}(\mu+\alpha) \pm \tfrac{1}{2}\sqrt{(\mu+\alpha)^2 - 4\mu(\alpha - \nu)} \\ &= \tfrac{1}{2s\gamma^\kappa}(\gamma^\kappa - \gamma) \pm \tfrac{1}{2}\sqrt{\tfrac{1}{s^2(\gamma^\kappa)^2}(\gamma^\kappa - \gamma)^2 - \kappa\tfrac{4}{s\ell}\gamma(1-\gamma)}. \end{aligned} \qquad [3.39]$$

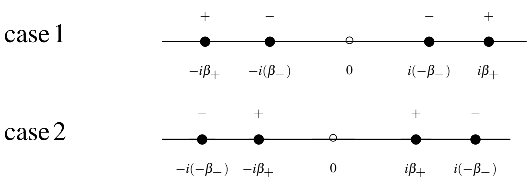
Clearly, both of the functions $\gamma \mapsto \beta_+$ and $\gamma \mapsto -\beta_-$ are continuous on $(0, \gamma_-) \cup (\gamma_+, 1)$, when $\kappa = +$, and on $(0,1)$, when $\kappa = -$. Moreover, they do not vanish. In fact, using an argument similar to the proof of lemma 3.6, it follows that both of these functions are strictly monotonic. From equation [3.39] we obtain

1) $\beta_+ > -\beta_- > 0$, if $\gamma \in (0, \gamma^-)$ and $\kappa = -$ or if $\gamma \in (0, \gamma_-)$ and $\kappa = +$;

2) $-\beta_- > \beta_+ > 0$, if $\gamma \in (\gamma^-, 1)$ and $\kappa = -$ or if $\gamma \in (\gamma_+, 1)$ and $\kappa = +$.

Since $X_\gamma|\Pi_\gamma^\pm = \begin{pmatrix} 0 & -\beta_\pm \\ \beta_\pm & 0 \end{pmatrix}$, the symplectic sign of X_γ on $\Pi_\gamma^\pm$ is $\operatorname{sgn}\beta_\pm$. In other words, the symplectic signs of X_γ on $\Pi_\gamma^\pm$ are $\pm$.

Listing the eigenvalues of X_γ in order of increasing absolute value and placing a sign above an eigenvalue corresponding to the symplectic sign of the corresponding symplectic plane spanned by the associated complex conjugate pair of purely imaginary eigenvalues, we obtain



Thus, we have proved the following.

CLAIM 5.2.– At $p = p_{1,-1}$, the matrix X_γ has symplectic signs $-$, $+$ when $\gamma \in (0, \gamma_-)$ and symplectic signs $+$, $-$ when $\gamma \in (\gamma_+, 1)$. At $p = p_{-1,-1}$, the matrix X_γ has symplectic signs $-$, $+$ when $\gamma \in (0, \gamma^-)$ and symplectic signs $+$, $-$ when $\gamma \in (\gamma^-, 1)$.

3.6. Bibliography

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