Chapter 2

WKB Solutions Near an Unstable Equilibrium and Applications

In this chapter, we present some precise results concerning spectral and scattering problems for the Schrödinger equation in the semi-classical regime, which we have obtained in a series of papers [ALE 08, BON 07, BON 11, BON]. As we can expect, properties of the underlying classical system play a crucial role in this regime, and we have studied the case where there exists one hyperbolic fixed point for the associated Hamiltonian flow. This occurs, for example, when the potential has a local maximum. Much is encoded in what we call a microlocal Cauchy problem at the fixed point, which we describe here in detail. In a physicist's language, the study of this microlocal Cauchy problem is that of the n-dimensional tunneling effect at the hyperbolic fixed point.

2.1. Introduction

In this chapter, we sum up different results obtained in a series of paper [BON 07, BON 11, BON, ALE 08] concerning spectral or scattering quantities attached to the semi-classical Schrödinger operator on $L^2(\mathbb{R}^n)$

$$P = -h^2\Delta + V(x), \qquad [2.1]$$

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and the corresponding classical Hamiltonian

$$p(x,\xi) = \sum_{j=1}^{n} \xi_j^2 + V(x). \qquad [2.2]$$

Here, h is a small positive parameter, $x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $V(x)$ is a real-valued smooth potential.

We suppose that $V(x)$ has a local non-degenerate maximum E_0 at a point, say at the origin $x = 0$. We investigate the asymptotic behavior as $h \to 0$ of solutions to the equation

$$Pu = Eu, \qquad [2.3]$$

when the spectral parameter E is in a vicinity of size $\mathscr{O}(h)$ of E_0. Of course, we are in a setting where the tunnel effect occurs at the barrier top. We will see quantitatively that, for such energies, tunneling governs the behavior of the physical quantities we are interested in.

Here, we have chosen to concentrate on a scattering situation, namely we assume that $E_0 > 0$ and $V(x) \to 0$ as $|x| \to +\infty$. In this setting, we will describe some results concerning resonances for the Schrödinger operator P.

In physics, the notion of quantum resonance appeared at the beginning of quantum mechanics. Its introduction was motivated by the behavior of various quantities related to scattering experiments, such as the scattering amplitude, the scattering cross-section or the time-delay (the derivative of the spectral shift function). At certain energies, these quantities present peaks (now called Breit–Wigner peaks), which were modelized by a Lorentzian-shaped function

$$w_{a,b} : \lambda \longmapsto \frac{1}{\pi} \frac{b}{(\lambda - a)^2 + b^2}.$$

The real numbers a and $(\pi b)^{-1} > 0$ are the location of the maximum of the peak and its height. The number $2b$ is the width of the peak (more precisely its width at half its height). Of course, for $\rho = a - ib \in \mathbb{C}$, we have

$$w_{a,b}(\lambda) = -\frac{1}{\pi} \frac{\operatorname{Im} \rho}{|\lambda - \rho|^2},$$

where the complex number ρ is a resonance. Such complex values for energies had also appeared, for example, in the work of Gamow [GAM 28], to explain α-radioactivity. In that context, the inverse of the imaginary part of the resonance appears to be the half-life time of the corresponding pseudo-particle.

On the mathematical side, the study of resonances for Schrödinger operators is more recent. It has permitted us to give a rigorous framework and obtain very precise results, in particular, on the location of resonances in relation to the geometry of the underlying classical flow. One of the most efficient mathematical definitions of resonances is based on the notion of complex scaling (see, e.g., [AGU 71, BAL 71, SIM 79, HUN 86, SIG 84, CYC 85, NAK 90, NAK 89, HEL 86, SJÖ 91]). As a matter of fact, resonances, both in the physical sense and in the mathematical sense, are poles in the lower half plane, say, of a suitable meromorphic extension of the resolvent $(P-E)^{-1}$ from the upper half plane through the essential spectrum of P (the positive real axis).

In a semi-classical regime, one expects, according to Bohr's correspondence principle, that the underlying classical system appears in the discussion. As a matter of fact, in our settings, classical quantities play the main role. Since works by Hörmander and others, the usual way to make the link between the quantum quantities and the classical quantities has been to use the language of microlocal analysis, here in the semi-classical setting. In particular, we will say that a function $u \in L^2(\mathbb{R}^n)$ is microlocally zero at a point (x_0, ξ_0) of the phase space, which means that there exists a smooth cut-off function χ, with $\chi(x_0, \xi_0) = 1$, such that

$$\chi^w(x, hD)u(x) = \mathscr{O}(h^\infty).$$

Here, $\chi^w(x, hD)$ is the semi-classical Weyl quantization of the cut-off function χ (see definition 2.1).

Of course, the key to the study of resonances is to have a good knowledge of the solutions u to the Schrödinger equation [2.3] for energies E close to the barrier top energy E_0, and more precisely their asymptotic Wentzel Kramers Brillouin (or WKB) behavior as $h \to 0$. In fact, the behavior of u outside of a compact set is rather clear since V is close to 0 there, and the main difficulty is to obtain a sharp enough description of u in a vicinity of the maximum point. More precisely, it appears that the microlocal behavior of u in a neighborhood of the hyperbolic fixed point in the phase space is the only thing that matters. The function $v = \chi^w u$, that is the function u truncated microlocally near the hyperbolic fixed point, satisfies the microlocal Cauchy problem

$$\begin{cases} Pv = Ev \text{ microlocally near the fixed point,} \\ v \text{ has a prescribed behavior in some incoming region.} \end{cases} \qquad [2.4]$$

This kind of microlocal Cauchy formulation is analogous to some normal form reduction, but can be used in more general geometric settings. Moreover, this approach avoids the use of an abstract reduction operator, and the solution of the problem [2.4] can be written explicitly. In the present case, the study of the microlocal Cauchy problem is given in section 2.2. The knowledge of the solution v of [2.4] allows us to obtain the asymptotic behavior of the solution u to [2.3], and, eventually, to compute the physical quantities we study.

Among the applications of this microlocal study, we focus here on the following two:

– Describe the behavior of the Schrödinger group, in the case where the potential V has the form of a single barrier of height E_0, for energies close to E_0. It turns out that the semi-classical expansion of this evolution operator involves resonances created by the barrier top. The results in section 2.2 are used to compute the non-orthogonal projection operator corresponding to each resonance, which appears in the representation formula of the evolution operator.

– Prove the existence of a resonance-free zone, i.e. give an estimate from below of the imaginary part of resonances, when the classical system possesses a homoclinic orbit. The results in section 2.2 along with the standard Maslov theory enable us to compute the decay of microlocal solution after a continuation along the homoclinic trajectories. This leads us to a contradiction if a resonance is assumed to be close enough to the real axis.

2.2. Connection of microlocal solutions near a hyperbolic fixed point

In this section, we have the following assumption:

(A1) $V(x)$ is a real-valued smooth function near the origin and the origin is a non-degenerate maximal point.

In suitable coordinates, the Taylor expansion at the origin can be written in the form

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + \mathscr{O}(x^3) \qquad \text{as } x \to 0, \qquad [2.5]$$

with maximal value E_0 and positive constants

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

2.2.1. *A model in one dimension*

In the remainder of this chapter, we will study the general n-dimensional case, but to start with, we recall here some well-known results concerning the simplest one-dimensional operator with such a hyperbolic fixed point. We study the asymptotic expansion of the solutions to the one-dimensional Schrödinger equation

$$Pu := \left(-h^2\frac{d^2}{dx^2} - \frac{\lambda^2}{4}x^2\right)u = hzu, \qquad [2.6]$$

with respect to the semi-classical parameter $h \to 0$. Here, λ is a positive constant and z is a spectral parameter bounded with respect to h. The potential $-\lambda^2x^2/4$ presents a non-degenerate barrier at $x = 0$ and the energy $E = hz$ is close to the maximum value $E_0 = 0$. In this one-dimensional simple case, we describe here the solutions in terms of Weber functions.

If $z = -i\lambda\left(k+\frac{1}{2}\right)$, $k \in \mathbb{N} := \{0,1,2,\ldots\}$, there exists a solution

$$u_k(x,h) = H_k\left(e^{-\pi i/4}\sqrt{\frac{\lambda}{h}}x\right)e^{i\lambda x^2/(4h)},$$

where H_k is the Hermite polynomial of degree k. The function u_k is an outgoing wave for $x \to \pm\infty$ (in the sense that its microsupport is included in the outgoing stable manifold of the corresponding classical Hamiltonian vector field, see sections 2.2.2 and 2.2.3).

If $z \in \mathbb{C}\setminus -i\lambda\left(\mathbb{N}+\frac{1}{2}\right)$, i.e. if $\nu := iz/\lambda - 1/2 \notin \mathbb{N}$, then

$$u_\nu(x,h) := D_\nu\left(e^{-\pi i/4}\sqrt{\frac{\lambda}{h}}x\right) \qquad [2.7]$$

is a solution to [2.6]. Here

$$D_\nu(y) = \frac{1}{\Gamma(-\nu)}\int_0^\infty \exp\left(-\left(\frac{y^2}{4} + y\eta + \frac{\eta^2}{2}\right)\right)\eta^{-\nu-1}d\eta,$$

is the Weber function. For any cut-off function χ that is identically equal to 1 on an interval $[0,R]$, we define

$$I_\nu(x,h) = \int_0^\infty \exp\left(\frac{i\lambda}{h}\left(\frac{x^2}{4} + x\xi + \frac{\xi^2}{2}\right)\right)\xi^{-\nu-1}\chi(\xi)\,d\xi. \qquad [2.8]$$

Then, we see the following.

PROPOSITION 2.1.–

1) $I_\nu(x,h)$ is a quasi-mode, i.e. for $|x|<R$, we have

$$(P-hz)I_\nu(x,h)=\mathscr{O}(h^\infty).$$

2) Solution [2.7] satisfies $u_\nu(x,h)=\text{const}\cdot I_\nu(x,h)+\mathscr{O}(h^\infty)$ on $L^2([-R,R])$.

3) Suppose ν stays in a compact subset of $\mathbb{C}\setminus\mathbb{N}$ for any h small enough. Then, I_ν has an asymptotic expansion in powers of h uniformly for x in any compact subset of $\mathbb{R}\setminus\{0\}$: for $x>0$, there exists a symbol $a(x,h)\sim\sum_{k=0}^\infty a_k(x)h^k$ with $a_0=1$ such that

$$I_\nu(x,h)=e^{-\pi i\nu/2}\Gamma(-\nu)\left(\frac{\lambda x}{h}\right)^\nu e^{i\lambda x^2/(4h)}a(x,h),$$

and, for $x<0$, there exist symbols $b(x,h)\sim\sum_{k=0}^\infty b_k(x)h^k$ with $b_0=1$ and $c(x,h)\sim\sum_{k=0}^\infty c_k(x)h^k$ with $c_0=1$ such that

$$\begin{aligned} I_\nu(x,h) &= e^{\pi i\nu/2}\Gamma(-\nu)\left(\frac{\lambda|x|}{h}\right)^\nu e^{i\lambda x^2/(4h)}b(x,h) \\ &\quad + e^{\pi i/4}\sqrt{\frac{2\pi h}{\lambda}}|x|^{-\nu-1}e^{-i\lambda x^2/(4h)}c(x,h). \end{aligned}$$

Here, $a(x,h)\sim\sum_{k=0}^\infty a_k(x)h^k$ means that for any $N\in\mathbb{N}$, $a(x,h)-\sum_{k=0}^N a_k(x)h^k=\mathscr{O}(h^{N+1})$.

The function $u_\nu(x,h)$ describes a wave coming from $x<0$ to the origin and scattered in the positive and negative directions. In the case of $z=0$, in particular, this proposition states that when the amplitude of the incoming wave is normalized to $|x|^{-1/2}$ then that of the transmitted wave in the region $x>0$ is $x^{-1/2}/\sqrt{2}$ and that of the reflected wave in the region $x<0$ is $|x|^{-1/2}/\sqrt{2}$ (see [2.33]).

However, in the case of $z=-i\lambda\left(k+\frac{1}{2}\right)$, the wave is purely outgoing. This means that for these energies, the incoming wave does not determine the outgoing wave.

In the following, we generalize this fact to the multidimensional case with potential having a non-degenerate local maximum. Theorem 2.2 guarantees that the incoming wave determines the outgoing wave except for a discrete set of energies, and theorem 2.3 provides the asymptotic behavior of the outgoing wave in terms of that of the incoming wave.

2.2.2. *Classical mechanics*

Recall that $p(x,\xi)$ is the classical Hamiltonian [2.2] with $V(x)$ satisfying (A1). Consider the canonical system of p

$$\frac{d}{dt}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \nabla_\xi p \\ -\nabla_x p \end{pmatrix}. \tag{2.9}$$

The origin $(x,\xi) = (0,0)$ is a fixed point of the Hamilton vector field H_p. The linearization of H_p at the origin is

$$\frac{d}{dt}\begin{pmatrix} x \\ \xi \end{pmatrix} = F_p \begin{pmatrix} x \\ \xi \end{pmatrix}, \tag{2.10}$$

where F_p is the fundamental matrix

$$F_p := \left.\begin{pmatrix} \frac{\partial^2 p}{\partial x \partial \xi} & \frac{\partial^2 p}{\partial \xi^2} \\ -\frac{\partial^2 p}{\partial x^2} & -\frac{\partial^2 p}{\partial \xi \partial x} \end{pmatrix}\right|_{(x,\xi)=(0,0)} = \begin{pmatrix} 0 & 2\mathrm{Id} \\ \frac{1}{2}\mathrm{diag}\,(\lambda_j^2) & 0 \end{pmatrix}.$$

This matrix has n positive eigenvalues $\{\lambda_j\}_{j=1}^n$ and n negative eigenvalues $\{-\lambda_j\}_{j=1}^n$. The eigenspaces $\Lambda_\pm^0$ corresponding to these positive and negative eigenvalues are, respectively, outgoing and incoming stable manifolds for the quadratic part p_0 of p:

$$\begin{aligned} \Lambda_\pm^0 &= \left\{(x,\xi) \in \mathbb{R}^{2n};\ \exp(tH_{p_0})(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\right\} \\ &= \left\{(x,\xi) \in \mathbb{R}^{2n};\ \xi_j = \pm\tfrac{\lambda_j}{2} x_j,\ j = 1,\ldots,n\right\}. \end{aligned}$$

By the stable manifold theorem, we also have outgoing and incoming stable manifolds for p:

$$\Lambda_\pm = \left\{(x,\xi) \in \mathbb{R}^{2n};\ \exp(tH_p)(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\right\}.$$

The tangent space of $\Lambda_\pm$ at $(0,0)$ is $\Lambda_\pm^0$. The manifolds $\Lambda_\pm$ are Lagrangian manifolds and can be written near $(0,0)$

$$\Lambda_\pm = \left\{(x,\xi) \in \mathbb{R}^{2n};\ \xi = \frac{\partial \phi_\pm}{\partial x}(x)\right\},$$

where the generating functions $\phi_\pm$ behave like

$$\phi_\pm(x) = \pm \sum_{j=1}^{n} \frac{\lambda_j}{4} x_j^2 + \mathscr{O}\big(|x|^3\big) \quad \text{as } x \to 0. \tag{2.11}$$

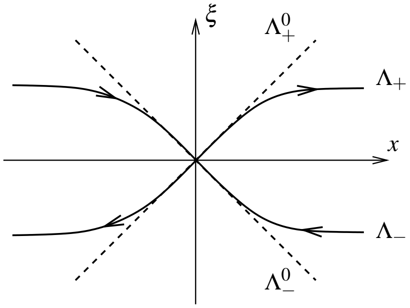

Figure 2.1. *The Lagrangian manifolds* $\Lambda_\pm$ *and* $\Lambda_\pm^0$

Now suppose $\rho_\pm = (x_\pm, \xi_\pm) \in \Lambda_\pm \setminus \{(0,0)\}$. Then by definition $\exp(tH_p)(\rho_\pm) \to (0,0)$ as $t \to \mp\infty$. More precisely, we have the following.

PROPOSITION 2.2.–

$$\exp(tH_p)(\rho_\pm) \sim \sum_{k=1}^{\infty} \gamma_k^\pm(t) e^{\pm\mu_k t} \quad \text{as } t \to \mp\infty,$$

where

$$0 < \mu_1 < \mu_2 < \cdots$$

are the linear combinations over $\mathbb{N}$ of $\{\lambda_j\}_{j=1}^n$, and, in particular, $\mu_1 = \lambda_1$. The $\gamma_k^\pm(t)$ are vector-valued polynomials in t. Moreover, γ_1 is independent of t and is an eigenvector of F_p corresponding to $\pm\lambda_1$. Note that $\gamma_1 e^{-\lambda_1 t}$ is a solution to [2.10].

In what follows, we will denote the x-space projection of the vector $\gamma_1^\pm(\rho_\pm)$ by $g_\pm(\rho_\pm)$.

REMARK 2.1.– By the symmetry with respect to ξ of $p(x,\xi)$, we have

$$\phi_-(x) = -\phi_+(x) \quad \text{and} \quad \Lambda_- = \big\{(x,-\xi) \in \mathbb{R}^{2n};\ (x,\xi) \in \Lambda_+\big\}.$$

If $\rho_\pm = (x, \pm\xi) \in \Lambda_\pm$, then

$$g_+(\rho_+) = g_-(\rho_-) =: g(x).$$

2.2.3. *Review of semi-classical microlocal analysis*

In this section, we recall some basic properties of the h-pseudo-differential calculus. For more details, we refer the reader to [DIM 99, MAR 02, ZWO 12]. We begin with the definition of the semi-classical pseudo-differential operators in Weyl quantization.

DEFINITION 2.1.– *Let $\chi(x,\xi)$ be a function in $C_b^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ (the space of functions bounded with all their derivatives). The pseudo-differential operator $\chi^w(x,hD)$ with symbol χ is defined by*

$$(\chi^w(x,hD)u)(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h} \chi\Big(\frac{x+y}{2},\xi\Big) u(y)\,dy\,d\xi,$$

for all u in the Schwartz space $S(\mathbb{R}^n)$.

In particular, if $\chi(x,\xi) = \chi(x)$ (respectively $\chi(x,\xi) = \chi(\xi)$), then $\chi^w(x,hD)$ is simply the multiplication operator by $\chi(x)$ (respectively the semi-classical Fourier multiplier by $\chi(\xi)$). We now define the notion of microsupport. Let $u(x;h)$ be in $L^2(\mathbb{R}^n)$ depending on h with $\|u\| \leq 1$ and (x_0,ξ_0) a point in the phase space $\mathbb{R}^{2n}$.

DEFINITION 2.2.– *We say that $u = 0$ microlocally at (x_0,ξ_0) if there exists a function $\chi \in C_0^\infty(\mathbb{R}^{2n})$ with $\chi(x_0,\xi_0) = 1$ such that*

$$\|\chi^w(x,hD)u\|_{L^2(\mathbb{R}^n)} = \mathscr{O}(h^\infty) \qquad \text{as } h \to 0. \tag{2.12}$$

The complement of the set of such points is called the microsupport (or frequency set).

In other words, $u = 0$ microlocally near (x_0,ξ_0) iff the function u does not oscillate near x_0 with semi-classical frequencies closed to ξ_0. If it is the case, then [2.12] holds true for all $\chi \in C_0^\infty(\mathbb{R}^{2n})$ supported in a neighborhood of (x_0,ξ_0). For $\Omega \subset \mathbb{R}^{2n}$, we say that $u = 0$ microlocally in Ω (respectively outside Ω) iff $u = 0$ microlocally near each point in Ω (respectively not in Ω). We recall now some fundamental properties of the microsupport.

PROPOSITION 2.3.– The microsupport of a function u is a closed set.

PROPOSITION 2.4.– Let $u(x;h) = a(x;h)e^{i\phi(x)/h}$, where $\phi(x)$ is a real-valued C^∞ function in a domain $\Omega \subset \mathbb{R}^n$ and $a(x;h)$ is a C^∞ symbol on Ω, i.e. $a(x;h)$ is bounded in Ω uniformly with respect to h with all its derivatives. Then

$$u = 0 \text{ microlocally outside } \Big\{(x,\xi) \in \mathbb{R}^{2n};\ \xi = \frac{\partial\phi}{\partial x}(x)\Big\}.$$

Eventually, we state the theorem of propagation of singularities that was first proved by Hörmander [HÖR 71] in the classical setting.

THEOREM 2.1.– PROPAGATION OF SINGULARITIES – Let u be a solution to [2.3] such that $\|u\| \leq 1$. The microsupport of u is included in the characteristic set. This means

$$u = 0 \text{ microlocally outside } \operatorname{Char}(p - E_0) := \big\{ (x,\xi) \in \mathbb{R}^{2n};\ p(x,\xi) = E_0 \big\}.$$

Moreover, for all $(x_0,\xi_0) \in \operatorname{Char}(p - E_0)$, $u = 0$ microlocally near $(x_0,\xi_0) \iff \forall t \in I, u = 0$ microlocally near $\exp(tH_p)(x_0,\xi_0)$, where $0 \in I$ is the maximal interval of existence of $\exp(tH_p)(x_0,\xi_0)$.

2.2.4. *The microlocal Cauchy problem – uniqueness*

In this section, we consider the microlocal Cauchy problem at a hyperbolic fixed point of the classical flow. As explained in section 2.1, this approach allows us to focus on the most important region of the phase space and, eventually, to obtain information on the global problem.

For a small neighborhood Ω of $(0,0)$ and $\varepsilon > 0$ small, we consider the microlocal Cauchy problem:

$$\begin{cases} Pu = Eu & \text{microlocally in } \Omega, \\ u = u_0(x) & \text{microlocally in } \mathscr{C} := \Lambda_- \cap \{|x| = \varepsilon\}, \end{cases} \qquad [2.13]$$

with $E = E_0 + hz$. Note that the initial surface $\mathscr{C}$ is transversal to the Hamilton flow for sufficiently small ε. Since we want to study quantities associated with the resonances that are non-real in general, the spectral parameter z may be complex but in a disc of center 0 and radius bounded with respect to h.

We start with a unique result for this problem. For the proof, we refer the reader to [BON 07, section 4]. Let r be any positive number and z be the complex number, which may depend on h, in a disc $D(r) := \{z \in \mathbb{C};\ |z| < r\}$.

THEOREM 2.2.– [BON 07, theorem 2.1]– There exists an h-independent positive number δ and an h-dependent finite set $\Gamma(h) \subset D(r) \cap \{z \in \mathbb{C};\ \operatorname{Im} z < -\delta\}$, whose cardinal number is bounded with respect to h, such that if $\operatorname{dist}(z, \Gamma(h)) > h^C$ for some $C > 0$, and if $u_0 = 0$, then any solution $u \in L^2(\mathbb{R}^n)$ of [2.13], satisfying $\|u\| \leq 1$, is 0 microlocally in a neighborhood Ω' of the origin.

REMARK 2.2.– In the analytic category (i.e. p is analytic near the origin and the notion of C^∞-microsupport (see definition 2.2) is replaced by the analytic microsupport (see [SJÖ 82])), we have the same theorem with more precision on the set $\Gamma(h)$. In fact, $\Gamma(h)$ is $-i\mathscr{E}_0$ modulo $\mathscr{O}(h)$, where

$$\mathscr{E}_0 = \Big\{ \sum_{j=1}^{n} \lambda_j \Big(\alpha_j + \frac{1}{2} \Big);\ (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \Big\}$$

is the set of eigenvalues of the harmonic oscillator

$$-\Delta + \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2, \qquad [2.14]$$

see [BON 07, theorem 2.2]. Note also that, modulo $o(h)$, $E_0 - ih\mathscr{E}_0$ is the set of the resonances generated by the barrier top (see theorem 2.5).

In the C^∞ case, Helffer and Sjöstrand [HEL 84] have constructed the asymptotic expansion (in powers of $h^{1/2}$) of the eigenvalues at the bottom of a potential well. The set of the first terms of the expansion is $\mathscr{E}_0$. This means that $-i\mathscr{E}_0$ is necessarily included in $\Gamma(h)$ modulo $\mathscr{O}(h)$. We expect that, modulo $\mathscr{O}(h^\infty)$, $\Gamma(h)$ is the set of $-i$ times the eigenvalues obtained in [HEL 84].

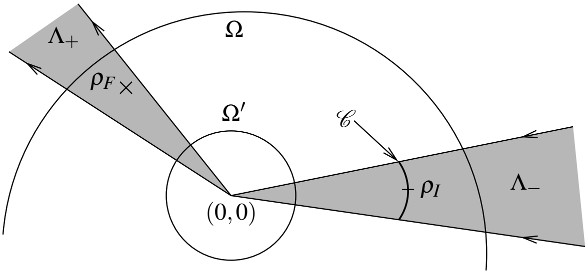

Figure 2.2. *The geometrical setting of theorems 2.2 and 2.3*

If $u = 0$ microlocally in Ω', it also vanishes microlocally in Λ_+ by theorem 2.1. Hence, this result can be expressed as follows: the microsupport propagates from the incoming stable manifold Λ_- to the outgoing stable manifold Λ_+ under a generic assumption on the energy z.

2.2.5. *The microlocal Cauchy problem – transition operator*

Theorem 2.2 states that the data u_0 given on $\Lambda_- \cap \{|x| = \varepsilon\}$ uniquely determines the solution u at any point $\rho_F = (x_F, \xi_F)$ on Λ_+ (if it exists). Our problem now is to construct u microlocally near ρ_F in terms of u_0 that, restricted to the initial surface $\mathscr{C}$, has its support in a small neighborhood of a point $\rho_I = (x_I, \xi_I) \in \mathscr{C}$.

We make two generic assumptions: one is on the spectral parameter z and the other is on the initial point $\rho_I = (x_I, \xi_I) \in \mathscr{C}$ and the final point $\rho_F = (x_F, \xi_F) \in \Lambda_+$:

(A2) There exists $\nu > 0$ such that $\operatorname{dist}(z, \Gamma(h)) > \nu$;

(A3) $g(x_I) \cdot g(x_F) \neq 0$.

In particular, $g(x_I) \neq 0$. This means that, in case $\lambda_1 < \lambda_2$, the Hamilton flow starting from ρ_I converges to the origin tangentially to the x_1-axis. In case $\lambda_1 = \lambda_2$, also, we can assume, without loss of generality, that the x_1-axis is parallel to $g(x_I)$. Since p is of real principal type near ρ_I, we can modify the initial surface $\mathscr{C}$ so that it is given by $\{x_1 = \varepsilon\} \cap \Lambda_-$ near ρ_I. Hence, denoting $x_I = (\varepsilon, x_I')$, the initial data u_0 on $\mathscr{C}$ is a function of x' in a small neighborhood of x_I' and 0 elsewhere.

THEOREM 2.3.– [BON 07, theorem 2.6]– Assume (A1), (A2) and (A3). The microlocal Cauchy problem [2.13] has a solution u (unique thanks to theorem 2.2). Microlocally near $\rho_F = (x_F, \xi_F)$, it has the following representation formula

$$u(x,h) = \frac{h^{S(z)/\lambda_1}}{(2\pi h)^{n/2}} \int_{\mathbb{R}^{n-1}} e^{i(\phi_+(x) - \phi_-(\varepsilon, y'))/h} d(x, y'; h) u_0(y')\, dy'. \qquad [2.15]$$

Here

$$S(z) = \frac{1}{2}\sum_{j=1}^{n} \lambda_j - iz, \qquad [2.16]$$

and the symbol $d \in S_h^0(1)$ has the following asymptotic expansion

$$d(x, y'; h) \sim \sum_{k=0}^{\infty} d_k(x, y', \ln h) h^{\widehat{\mu}_k/\lambda_1}, \qquad [2.17]$$

where $0 = \widehat{\mu}_0 < \widehat{\mu}_1 (= \mu_2 - \mu_1) < \widehat{\mu}_2 < \cdots$ is a numbering of the linear combinations of $\{\mu_k - \mu_1\}_{k=0}^{\infty}$ over $\mathbb{N}$, and $d_k(x, y', \ln h)$ are polynomials in $\ln h$. In particular, the symbol d_0 is independent of $\ln h$.

We will need an explicit expression of the principal term d_0 of the symbol d for theorem 2.10, especially for the definition of $\mathscr{J}_0(\alpha)$ in [2.31]. It is given by

$$\begin{aligned} d_0(x,y') &= e^{-i\pi n/4}\lambda_1^{1/2-S(z)/\lambda_1}\exp\left(-\frac{S(z)}{2\lambda_1}\pi i\sigma\right)\Gamma\left(\frac{S(z)}{\lambda_1}\right) \\ &\quad\times e^{I_\infty(x)}\sqrt{\frac{|\det\nabla^2_{y'y'}\phi_-(\varepsilon,y')|}{J_\infty(y')}}\frac{|g(\varepsilon,y')|}{|g(\varepsilon,y')\cdot g(x)|^{\frac{S(z)}{\lambda_1}}}. \end{aligned} \qquad [2.18]$$

Here, $\sigma = \operatorname{sgn}(g(x_I)\cdot g(x_F))$,

$$I_\infty(x) := \int_0^{-\infty}\left(\Delta\phi_+(x(\tau)) - \frac{1}{2}\sum_{j=1}^{n}\lambda_j\right)d\tau,$$

where $x(t)$ is the x-space projection of the flow $\exp(tH_p)(\rho_F)$, and

$$\begin{aligned} J(t,y',\eta') &:= \det\frac{\partial x(t,y',\eta')}{\partial(t,y')}, \\ J_\infty(y') &:= \lim_{t\to+\infty}\frac{J(t,y',\eta')}{J(0,y',\eta')}\Big|_{\eta'=\frac{\partial\phi_-}{\partial y'}(\varepsilon,y')} e^{(-\sum_{j=1}^{n}\lambda_j+2\lambda_1)t}, \end{aligned}$$

where $x(t,y',\eta')$ is the x-space projection of the flow $\exp(tH_p)\rho(y',\eta')$ for y' near x'_I and η' near ξ'_I, and $\rho(y',\eta') := \left(\varepsilon,y';-\sqrt{-|\eta'|^2-V(\varepsilon,y')},\eta'\right) \in \{x_1 = \varepsilon\} \cap p^{-1}(E_0)$.

The main idea of the proof for theorem 2.3 is to express the solution u microlocally near the fixed point $(0,0)$ as a superposition of WKB solutions to the time-dependent Schrödinger equation

$$u(x,h) = \frac{1}{\sqrt{2\pi h}}\int_0^\infty e^{i\varphi(t,x)/h}a(t,x;h)\,dt.$$

Then, the phase $\varphi(t,x)$ has an asymptotic expansion as $t \to +\infty$

$$\varphi(t,x) \sim \phi_+(x) + \sum_{k=1}^{\infty}\phi_{\mu_k}(t,x)e^{-\mu_k t},$$

and the symbol $a(t,x;h)$ has classical expansion in h

$$a(t,x;h) \sim \sum_{\ell=0}^{\infty} a_\ell(t,x) h^\ell,$$

whose coefficients have expansion as $t \to +\infty$

$$a_\ell(t,x) \sim \sum_{k=0}^{\infty} a_{\ell,k}(t,x) e^{-(S+\mu_k)t},$$

where $a_{\ell,k}(t,x)$ is polynomial in t and $S = S(z)$ is defined by [2.16]. In particular, $a_{0,0}$ can be explicitly calculated from the initial condition on Λ_- and gives the value of the symbol d_0 on Λ_+.

2.3. Applications to semi-classical resonances

We first recall the definition of the resonances by the complex scaling method (see [AGU 71, HUN 86, SJÖ 91] and the other references given in section 2.1). This technique is very efficient in the semi-classical setting since it is well adapted to the microlocal calculus and since the resonances are seen as the (usual) eigenvalues of a non-self-adjoint operator. There exist other approaches to define the resonances (poles of different scattering quantities (see [LAX 89]), poles of the extension of the cut-off resolvent (see [2.20]), etc.). In fact, all these definitions coincide as proved in [HEL 87].

To define resonances, we assume

(B1) $V(x) \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and extends holomorphically in a sector

$$\mathscr{S} = \big\{ x \in \mathbb{C}^n;\ |\operatorname{Im} x| \leq (\tan\theta_0)|\operatorname{Re} x| \text{ and } |\operatorname{Re} x| > C \big\},$$

for some positive constants θ_0 and C. Moreover

$$V(x) \longrightarrow 0 \quad \text{as } |x| \to \infty \text{ in } \mathscr{S}.$$

Then, P is a self-adjoint operator on $L^2(\mathbb{R}^n)$ with $\sigma_{ess}(P) = \mathbb{R}_+$. To this operator, we associate a *distorted* operator

$$\widetilde{P}_\mu = U_\mu P U_{-\mu}, \quad (U_\mu f)(x) := |\det(\mathrm{Id} + \mu dF)|^{1/2} f(x + \mu F(x)),$$

for small real μ and $F \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with

$$F(x) = 0 \text{ on } |x| < R \quad \text{and} \quad F(x) = x \text{ on } |x| > R+1,$$

for large R. This operator $\widetilde{P}_\mu$ is analytic of type-A with respect to μ, and, taking R large enough, $P_\theta := \widetilde{P}_{i\theta}$ is well defined for θ small enough. Then, $\sigma_{ess}(P_\theta) = e^{-2i\theta}\mathbb{R}_+$ and the spectrum of P_θ in $C_\theta := \{E \in \mathbb{C}\backslash\{0\};\ -2\theta < \arg E < 0\}$ is discrete.

DEFINITION 2.3.– *Resonances are the eigenvalues of P_θ in C_θ. The multiplicity of a resonance E^* is the rank of the spectral projection*

$$\Pi_{E^*} = \frac{1}{2\pi i}\int_\gamma (E - P_\theta)^{-1} dE, \qquad [2.19]$$

where γ is a small circle centered at E^* and we choose θ with $E^* \in C_\theta$. Resonances are independent of θ in the sense that $\sigma(P_{\theta'}) \cap C_\theta = \sigma(P_\theta) \cap C_\theta$ for $\theta < \theta'$ by taking into account the multiplicity. Moreover, the resonances are also independent of F. Hence, we will denote the set of resonances by $\operatorname{Res}(h)$ *without indicating θ and F.*

Equivalently, we can define the resonances of P by showing that the resolvent $(E-P)^{-1} : L^2_{comp}(\mathbb{R}^n) \longrightarrow L^2_{loc}(\mathbb{R}^n)$ has a meromorphic extension $R_+(E)$ from the upper half plane to C_θ across $(0,\infty)$. We have

$$\chi R_+(E)\chi = \chi(E - P_\theta)^{-1}\chi. \qquad [2.20]$$

for any cut-off function χ whose support is in $|x| < R$. The poles are the resonances and the multiplicity of a resonance is also given by rank $\frac{1}{2\pi i}\int_\gamma R_+(E)dE$.

Let $K(E)$ be the set of trapped trajectories on the energy surface $p^{-1}(E)$:

$$K(E) = \big\{(x,\xi) \in p^{-1}(E);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\big\}.$$

The following result suggests a close relationship between the semi-classical distribution of resonances near a real energy E and the geometry of $K(E)$ of the corresponding classical dynamics.

THEOREM 2.4.– [MAR 02a] Let $E_0 > 0$ be such that $K(E_0) = \emptyset$. Then, there exists $\varepsilon > 0$ such that, for any $C > 0$, there is no resonance in the box

$$[E_0 - \varepsilon, E_0 + \varepsilon] + i\big[-Ch|\ln h|, 0\big],$$

for sufficiently small h.

In the case where $V(x)$ is globally analytic near $\mathbb{R}^n$, it was earlier proved by Helffer and Sjöstrand [HEL 85] and also by Briet *et al.* [BRI 87a] under a stronger hypothesis called the virial assumption that there is no resonance in an h-independent neighborhood of E_0 such that $K(E_0) = \emptyset$.

In sections 2.3.1 and 2.3.2, we assume (A1) and (B1). The maximal value E_0 at the origin should then be positive. $K(E_0)$ contains at least the point $(0,0)$ and we consider resonances close to E_0.

2.3.1. *Spectral projection and Schrödinger group*

Under (A1), the origin $(0,0)$ is a hyperbolic fixed point and is itself a trapped point in $p^{-1}(E_0)$. Here, we study the case where it is the only trapped point, i.e.

(B2) $K(E_0) = \{(0,0)\}$.

This assumption implies that E_0 is the global maximum of V and it is attained only at $x = 0$.

When $V(x)$ is assumed to be analytic globally near $\mathbb{R}^n$, the semi-classical distribution of resonances is known near the barrier top energy E_0 (in [BRI 87b], a virial condition is assumed).

THEOREM 2.5.– [BRI 87b, SJÖ 87]– Let $\mathrm{Res}_0(h)$ be the discrete set

$$\mathrm{Res}_0(h) := E_0 - ih\mathscr{E}_0 = \left\{ E_\alpha^0 := E_0 - ih\sum_{j=1}^{n} \lambda_j \left(\alpha_j + \frac{1}{2}\right);\ \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\},$$

and let C be an h-independent positive constant such that $C \neq \sum_{j=1}^{n} \lambda_j\ (\alpha_j + \frac{1}{2})$ for any $\alpha \in \mathbb{N}^n$. Then, in $D(E_0, Ch)$, there exists a bijection

$$b_h : \mathrm{Res}_0(h) \cap D(E_0, Ch) \longrightarrow \mathrm{Res}(h) \cap D(E_0, Ch),$$

such that $b_h(E) = E + o(h)$.

Let us denote $E_\alpha = b_h(E_\alpha^0)$. We call E_α^0 *pseudo-resonance* (see [SJÖ 01]). We say that a pseudo-resonance E_α^0 is *simple* if $E_\alpha^0 = E_{\alpha'}^0$ implies $\alpha = \alpha'$. If a pseudo-resonance E_α^0 is simple, then the corresponding resonance E_α is simple for h small enough (i.e. its multiplicity is one), and has an asymptotic expansion in powers of h whose leading term is E_α^0.

THEOREM 2.6.– [BON 11, theorem 4.1]– Assume (A1), (B1), (B2) and suppose $E_\alpha^0 \in \mathrm{Res}_0(h)$ is simple. Then, as operator from $L^2_{comp}(\mathbb{R}^n)$ to $L^2_{loc}(\mathbb{R}^n)$, we have

$$\Pi_{E_\alpha} = c(h)(\cdot, \overline{f_\alpha}) f_\alpha, \tag{2.21}$$

with

$$c(h) = h^{-|\alpha|-\frac{n}{2}} \frac{e^{-i\frac{\pi}{2}(|\alpha|+\frac{n}{2})}}{(2\pi)^{\frac{n}{2}}\alpha!} \prod_{j=1}^{n} \lambda_j^{\alpha_j+\frac{1}{2}}, \tag{2.22}$$

where $f_\alpha = f_\alpha(x,h)$ is a solution to $Pf_\alpha = E_\alpha f_\alpha$, locally L^2 uniformly in h, vanishes in the incoming region (in the microlocal sense) and has an asymptotic expansion as $h \to 0$ for x near the origin

$$f_\alpha = d_\alpha(x,h) e^{i\phi_+(x)/h}, \tag{2.23}$$

with

$$d_\alpha(x,h) \sim \sum d_{\alpha,j}(x) h^j \quad \text{as } h \to 0, \tag{2.24}$$

$$d_{\alpha,0}(x) = x^\alpha + \mathscr{O}(|x|^{|\alpha|+1}) \quad \text{as } x \to 0. \tag{2.25}$$

The proof of theorem 2.6 is as follows. First, we choose a suitable u_0 (a Lagrangian distribution which associated Lagrangian manifold that is transverse to Λ_-). Then, we compute the solution of the Cauchy problem [2.13] for E close to E_α using theorem 2.3. Performing the integration in E around E_α as in [2.19], we compute the asymptotic of $\Pi_{E_\alpha} u_0$. The leading term with respect to h comes from the singularity of the function Γ in [2.18]. In particular, this gives all the stated properties for f_α. Finally, the coefficient $c(h)$ follows from the computation of $(u_0, \overline{f_\alpha})$.

Let us consider the Cauchy problem for the time-dependent Schrödinger equation

$$\begin{cases} ih\dfrac{\partial \psi}{\partial t}(t,x) = P\psi(t,x), \\ \psi(0,x) = \psi_0(x). \end{cases}$$

We denote the solution $\psi(t,x)$ by $e^{-itP/h}\psi_0$. The operator $e^{-itP/h}$ is unitary on $L^2(\mathbb{R}^n)$.

Recall that if E^* is an isolated eigenvalue of P, then for any $\psi(E) \in C_0^\infty(\mathbb{R})$ supported near E^*, we have

$$e^{-itP/h}\psi(P) = e^{-itE^*/h}\Pi_{E^*}\psi(E^*),$$

where Π_{E^*} is the orthogonal projection to the eigenspace of E^* generated by orthonormal eigenfunctions $\{f_j\}$,

$$\Pi_{E^*} = \sum_j (\cdot, f_j) f_j.$$

In the case of resonances associated with a single barrier top, we have, using the projection operator of the previous theorem, the following theorem.

THEOREM 2.7.– [BON 11, theorem 6.1]– Assume (A1), (B1), (B2). Let C be any positive constant such that $C \neq \sum_{j=1}^n (\beta_j + \frac{1}{2})\lambda_j$ for all $\beta \in \mathbb{N}^n$. Then, for any $\chi \in C_0^\infty(\mathbb{R}^n)$ and any $\psi \in C_0^\infty(\mathbb{R})$ supported in a sufficiently small neighborhood of E_0, there exists $K > 0$ such that for any t, we have as $h \to 0$,

$$\begin{aligned}\chi e^{-itP/h}\chi\psi(P) = \sum_{E_\alpha \in \mathrm{Res}(h)\cap D(E_0,Ch)} & \chi \,\mathrm{Residue}_{E_\alpha}\left(e^{-itE/h}R_+(z)\right)\chi\,\psi(P) \\ & + \mathscr{O}(h^\infty) + \mathscr{O}(e^{-Ct}h^{-K}).\end{aligned} \quad [2.26]$$

If, in particular, all the pseudo-resonances in $D(E_0, Ch)$ are simple, we have, for any t, and as $h \to 0$,

$$\begin{aligned}\chi e^{-itP/h}\chi\psi(P) = \sum_{E_\alpha \in \mathrm{Res}(h)\cap D(E_0,Ch)} & e^{-itE_\alpha/h}\chi\Pi_{E_\alpha}\chi\,\psi(P) \\ & + \mathscr{O}(h^\infty) + \mathscr{O}(e^{-Ct}h^{-K}).\end{aligned} \quad [2.27]$$

Here, Π_{E_α} is the spectral projection given by [2.19].

REMARK 2.3.– We see in theorem 2.6 that $\chi\Pi_{E_\alpha}\chi \sim h^{-|\alpha|-n/2}$ when E_α^0 is simple. Since, on the other hand, $|e^{-itE_\alpha/h}| = e^{-t|\mathrm{Im}E_\alpha|/h} \sim e^{-t\sum_{j=1}^n \lambda_j(\alpha_j+\frac{1}{2})}$ for $E_\alpha \in \mathrm{Res}(h) \cap D(E_0, Ch)$, the α-th term of the RHS of [2.27] is greater than the errors for

$$t \geq \frac{K - \frac{n}{2} - |\alpha|}{C - \sum_{j=1}^n \lambda_j(\alpha_j + \frac{1}{2})} \ln\frac{1}{h} + \mathrm{const.} \quad [2.28]$$

REMARK 2.4.– If $\{\lambda_j\}_{j=1}^n$ are $\mathbb{Z}$-independent, all the pseudo-resonances are simple and [2.27] holds for any C.

2.3.2. *Resonance-free zone for homoclinic trajectories*

Here, we assume, instead of (B2), that $K(E_0)$ consists of the fixed point $(0,0)$ and homoclinic trajectories associated with this point. More precisely,

(B3) $K(E_0) = \Lambda_+ \cap \Lambda_-$ and $\mathscr{H} := \Lambda_+ \cap \Lambda_- \setminus \{(0,0)\} \neq \emptyset$.

This is the case when there is another suitably shaped bump higher than E_0. Note that there may be infinitely many homoclinic trajectories (see example 2.2).

When the dimension is 1 and the potential is analytic, the operator $P - E$ can be reduced microlocally near $(0,0)$ to the Weber equation [2.6], see [HEL 89]. This fact combined with the complex WKB method lead us to the following result:

THEOREM 2.8.– [FUJ 98, theorem 0.7]– Assume $n = 1$, (A1), (B1), (B3), $\mathscr{H}$ consists of a unique curve and $V(x)$ is globally analytic near $\mathbb{R}$. Then, the resonances in the disc centered at E_0 with radius $Ch/|\ln h|$ with $C > 0$ satisfy

$$E_k = E_0 - \lambda_1 \frac{S_0 - (2k+1)\pi h}{|\ln h|} - i\frac{\ln 2}{2}\lambda_1 \frac{h}{|\ln h|} + \mathscr{O}(h/|\ln h|^2),$$

where $S_0 = \int_{\mathscr{H}} \xi \cdot dx$ is the action along the homoclinic curve $\mathscr{H}$ and $k \in \mathbb{N}$. In particular,

$$\operatorname{Im} E_k = -\frac{\ln 2}{2}\lambda_1 \frac{h}{|\ln h|} + \mathscr{O}(h/|\ln h|^2).$$

Let us consider the multidimensional case. To apply theorem 2.3, we need an assumption corresponding to (A3):

(B4) $g(x) \cdot g(x') \neq 0$ for any $x, x' \in \Pi_x \mathscr{H}$.

When there is only one homoclinic trajectory, this condition requires that the homoclinic trajectory should reach the barrier top in the direction of the minimum curvature. When the barrier top is isometric and there are many homoclinic trajectories as in example 3.14, this condition requires $\theta_1 < \pi/4$.

We will see that the imaginary part of resonances depends on the "strength" of the trap. We start with a case where the trapping is weak:

(B5) Either (B5)a) or (B5)b) holds:

a) $\lambda_1 < \lambda_n$;

b) $\forall \rho \in \mathscr{H}, \quad T_\rho \Lambda_+ \neq T_\rho \Lambda_-$.

Assumption (B5)a) means that $\mathscr{H}$ is small near the fixed point $(0,0)$ in the sense that x-space projection of every Hamilton curve in $\mathscr{H}$ is tangent to a subspace of $T_0\mathbb{R}^n$ of dimension $\leq n-1$.

THEOREM 2.9.– [BON]– Assume (A1), (B1), (B3), (B4) and (B5). Then, there exists $\delta > 0$ such that, for all $C > 0$, P has no resonance in

$$[E_0 - Ch, E_0 + Ch] + i[-\delta h, 0], \qquad [2.29]$$

for sufficiently small h. Moreover, for all $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists $M > 0$ such that for any E in this domain, we have

$$\left\| \chi (E-P)^{-1} \chi \right\| \lesssim h^{-M}.$$

Next, we consider the complementary case where the trapping is strong. We assume an isotropic condition on the barrier top:

(B6) $\lambda_1 = \cdots = \lambda_n =: \lambda$.

In this special setting, proposition 2.2 about the Hamiltonian flow on $\Lambda_\pm$ can be expressed as follows:

LEMMA 2.1.– For any $\alpha \in \mathbb{S}^{n-1}$, there exists a unique Hamiltonian curve $\rho_+(t,\alpha) = (x_+(t,\alpha), \xi_+(t,\alpha))$ on Λ_+ such that, for any $\varepsilon > 0$,

$$x_+(t,\alpha) = e^{\lambda t}\alpha + \mathscr{O}\left(e^{(2\lambda - \varepsilon)t}\right) \quad \text{as } t \to -\infty.$$

Then, we define

$$\mathscr{H}_{\text{tang}} := \{\rho \in \mathscr{H};\ T_\rho \Lambda_+ = T_\rho \Lambda_-\},$$

the set of the points at which Λ_+ and Λ_- are tangent, and

$$\begin{aligned} \mathscr{H}^\infty &:= \{\alpha \in \mathbb{S}^{n-1};\ \rho_+(\cdot,\alpha) \in \mathscr{H}\}, \\ \mathscr{H}^\infty_{\text{tang}} &:= \{\alpha \in \mathbb{S}^{n-1};\ \rho_+(\cdot,\alpha) \in \mathscr{H}_{\text{tang}}\}, \end{aligned}$$

the asymptotic directions of the Hamiltonian curves in $\mathscr{H}$ and $\mathscr{H}_{\text{tang}}$. Note that these two sets are compact subsets of $\mathbb{S}^{n-1}$.

Let $\alpha \in \mathscr{H}^\infty_{\mathrm{tang}}$. For any sufficiently small $\varepsilon > 0$, there exist unique times $t^\varepsilon_\pm(\alpha)$ satisfying $|x_+(t^\varepsilon_\pm(\alpha),\alpha)| = \varepsilon$ and $t^\varepsilon_\pm(\alpha) \to \mp\infty$ as $\varepsilon \to 0$. Then, it is well known that the quantity

$$\mathscr{M}_\varepsilon(\alpha) = \frac{\mathscr{D}(t^\varepsilon_+(\alpha),\alpha)}{\mathscr{D}(t^\varepsilon_-(\alpha),\alpha)} \quad \text{with} \quad \mathscr{D}(t,\alpha) = \sqrt{\left|\det \frac{\partial x_+(t,\alpha)}{\partial(t,\alpha)}\right|},$$

represents the evolution of the amplitude of WKB solutions along the curve $x_+(t,\alpha)$ from the time $t^\varepsilon_+(\alpha)$ to the time $t^\varepsilon_-(\alpha)$ (see, for example, [MAS 81]). This function $\mathscr{M}_\varepsilon(\alpha)$ has a positive limit $\mathscr{M}_0(\alpha)$ as ε tends to 0

$$\mathscr{M}_0(\alpha) := \lim_{\varepsilon\to 0} \mathscr{M}_\varepsilon(\alpha), \qquad [2.30]$$

which is continuous with respect to $\alpha \in \mathscr{H}^\infty_{\mathrm{tang}}$ and hence bounded. We also define a constant associated with the quantum propagation through the fixed point:

$$\mathscr{J}_0(\alpha) := (2\pi)^{-n/2}\Gamma\Big(\frac{n}{2}\Big)\int_{\mathscr{H}^\infty_{\mathrm{tang}}} |\alpha\cdot\omega|^{-n/2} d\omega. \qquad [2.31]$$

The amplification around the trapped set is then controlled by the quantity

$$\mathscr{A}_0 := \max_{\alpha\in\mathscr{H}^\infty_{\mathrm{tang}}} \mathscr{M}_0(\alpha)\mathscr{J}_0(\alpha) \in [0,+\infty[. \qquad [2.32]$$

REMARK 2.5.– In the one-dimensional case, $\mathscr{H}^\infty = \mathscr{H}^\infty_{\mathrm{tang}} \subset \{-1,1\}$ and, for each $\alpha \in \mathscr{H}^\infty$, we have

$$\mathscr{M}_0(\alpha) = 1, \quad \mathscr{J}_0(\alpha) = \begin{cases} 0 & \text{if } \mathscr{H}^\infty = \emptyset, \\ 1/\sqrt{2} & \text{if } \mathscr{H}^\infty = \{1\} \text{ or } \{-1\}, \\ \sqrt{2} & \text{if } \mathscr{H}^\infty = \{-1,1\}. \end{cases} \qquad [2.33]$$

THEOREM 2.10.– [BON]– Assume (A1), (B1), (B3), (B4), (B6) and

$$\mathscr{A}_0 < 1. \qquad [2.34]$$

Then, for all $\varepsilon > 0$, there exists $\nu > 0$ such that P has no resonance in the box

$$[E_0 - \nu h, E_0 + \nu h] + i\left[\big(\lambda \ln \mathscr{A}_0 + \varepsilon\big)\frac{h}{|\ln h|}, 0\right], \qquad [2.35]$$

for sufficiently small h. Moreover, for all $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists a positive constant M such that, for any E in this domain, we have

$$\|\chi(P-E)^{-1}\chi\| \lesssim h^{-M}, \tag{2.36}$$

for sufficiently small h.

When $\mathscr{A}_0 = 0$, we use the convention that $\ln(\mathscr{A}_0)$ appearing in [2.35] can be taken as any arbitrary large negative constant. We refer to [BON] for a result in a larger zone.

EXAMPLE 2.1.– Consider the case $n = 1$. Because of remark 2.5, condition [2.34] is satisfied if $\mathscr{H}^\infty$ consists of one point but not satisfied if $\mathscr{H}^\infty = \{-1, 1\}$. When $\mathscr{H}^\infty = \{1\}$ or $\mathscr{H}^\infty = \{-1\}$, the precise location of the resonances is given in theorem 2.8. This result implies that our estimate [2.35] from below the imaginary part of the resonances is optimal. When $\mathscr{H}^\infty = \{-1, 1\}$, on the contrary, we are in the "well in an island" situation, and the resonances are exponentially close to the real axis.

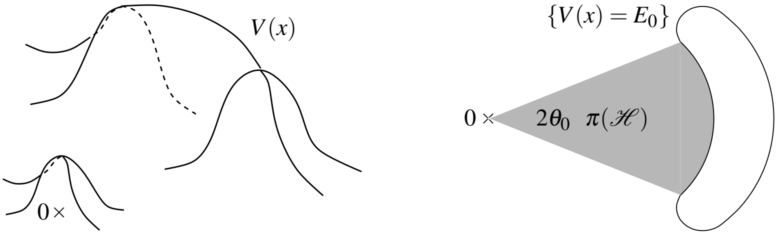

Figure 2.3. *The potential of example 2.1 and the spatial projection of* $\mathscr{H}$

EXAMPLE 2.2.– In dimension $n = 2$, let (r, θ) be the polar coordinates. We consider

$$V(x) = q_0(r) + q_1(r-a)\psi(\theta),$$

where the $q_\bullet(r)$'s are even non-degenerate bumps in $C_0^\infty(\mathbb{R})$ with $rq_\bullet'(r) < 0$ for $r \neq 0$ and $E_0 = q_0(0) < q_1(0)$, a is a sufficiently large constant such that $\operatorname{supp} q_0(r) \cap \operatorname{supp} q_1(r-a) = \emptyset$ and $\psi(\theta) \in C_0^\infty([-\theta_1 - \varepsilon, \theta_1 + \varepsilon])$ is equal to 1 for $|\theta| \leq \theta_1$ and $\theta\psi'(\theta) < 0$ for $\theta_1 < |\theta| < \theta_1 + \varepsilon$ for $\theta_1 < \pi/4$ and small enough $\varepsilon > 0$. The setting is illustrated in Figure 2.3. It can be checked that conditions (A1), (B1), (B3), (B4) and (B6) are all satisfied, and, moreover, $\mathscr{H}^\infty = \mathscr{H}^\infty_{\text{tang}} = [-\theta_1, \theta_1]$ and $\mathscr{M}_0(\alpha) = 1$. $\mathscr{J}_0(\alpha)$ can also be computed explicitly, and condition [2.34] is satisfied if $\sin(2\theta_1) < \tanh(2\pi)$.

We conclude this review by sketching the proofs of theorems 2.9 and 2.10. For the detail, we refer to [BON]. Assuming that there existed a resonance in the expected resonance-free domain [2.29] or [2.35], we would conclude that the corresponding normalized resonant state becomes smaller microlocally at any point on $\mathscr{H}$ after a continuation along homoclinic trajectories and the fixed point.

For the continuation along the homoclinic trajectories, we use the standard WKB theory of Maslov, which states, in particular, that the order in h of the amplitude of WKB solutions does not change along Hamiltonian flow.

For the continuation through the fixed point, we apply the results in section 2.2. We first show that the resonant state has its microsupport only on Λ_+. In particular, this implies that it is microlocally 0 on Λ_- outside $\mathscr{H}$. Hence, theorem 2.5 gives us its asymptotic behavior on $\Lambda_+ \cap \mathscr{H}$ from the knowledge of that on $\Lambda_+ \cap \mathscr{H}$. In the case of theorem 2.9, the amplitude of the resonant state changes by multiplication by h^α for some $\alpha > 0$, which comes from the prefactor $h^{S(z)/\lambda_1}$ in [2.15] when (B5)a) holds and from a stationary phase expansion of the integral in [2.15] when (B5)b) holds. In the case of theorem 2.10, the amplitude changes only by the multiplication by a small constant independent of h, therefore we need the explicit expression [2.18] of the principal symbol.

2.4. Acknowledgment

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