Chapter 18

Unfreezing Casimir Invariants: Singular Perturbations Giving Rise to Forbidden Instabilities

The infinite-dimensional mechanics of a fluid or a plasma can be formulated as a "noncanonical" Hamiltonian system on a phase space of Eulerian variables. Singularities of the Poisson bracket operator produce singular Casimir elements that foliate the phase space, imposing topological constraints on the dynamics. Here, we proffer a physical interpretation of Casimir elements as *adiabatic invariants* – upon coarse graining microscopic angle variables, we obtain a macroscopic hierarchy on which the separated action variables become adiabatic invariants. On reflection, a Casimir element may be *unfrozen* by recovering a corresponding angle variable; such an increase in the number of degrees of freedom is, then, formulated as a *singular perturbation*. As an example, we propose a canonization of the resonant singularity of the Poisson bracket operator of the linearized magnetohydrodynamics (MHD) equations, by which the ideal obstacle (resonant Casimir element) constraining the dynamics is unfrozen, giving rise to a tearing-mode instability.

18.1. Introduction

Although canonical Hamiltonian mechanics is described by a Poisson bracket operator (field tensor) that has a full rank on a symplectic manifold, general *noncanonical* Hamiltonian mechanics is endowed with a Poisson bracket operator (henceforth Poisson operator) that may have a non-trivial kernel; the corresponding

Chapter written by Zensho YOSHIDA and Philip J. MORRISON.

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Poisson manifold may then be split into some local symplectic leaves (Lie–Darboux theorem). A Casimir element foliates the Poisson manifold (with the gradient of a Casimir element belonging to the kernel of the Poisson operator)[1]. Consequently, an orbit is constrained to a leaf (level set) of a Casimir element, i.e. a Casimir element is a constant of motion. The constancy of a Casimir element is independent of the Hamiltonian (whereas a usual constant of motion pertains to some symmetry of a Hamiltonian), and it is due to a singularity of the Poisson operator. Here, we proffer an interpretation: "a Casimir element is an *adiabatic invariant* that is separated from a microscopic angle variable by coarse graining" – a Casimir leaf is then a *macroscopic hierarchy*. On reflection, a Casimir invariant may be *unfrozen* by recovering a corresponding angle variable. Such an increase in the number of degrees of freedom is, then, formulated as a *singular perturbation* (see [YOS 10]).

We will cast the theory of *tearing-mode instabilities* into a perspective of an unfrozen Casimir leaf. The dynamics of an ideal plasma obeys topological constraints on the co-moving magnetic field[2], which prevents the onset of various potential instabilities. A "tearing mode" is a mode of deformation that brings about a change in the topology of magnetic surfaces (the integral surfaces of magnetic field lines), which is, therefore, forbidden to occur in the ideal dynamics [FUR 63a, FUR 63b, WHI 83]. In [YOS 12], a tearing mode is formulated as an equilibrium point on a helical-flux Casimir leaf. As long as the helical-flux Casimir is constrained, the tearing mode cannot grow. We will formulate a "perturbation" that allows the system to change the helical flux, as well as absorb (dissipate) the energy. An unstable tearing mode has negative energy, with respect to a fiducial "helical" bifurcated equilibrium; hence, a tearing-mode instability is a process of moving across Casimir leaves toward a lower energy equilibrium point with a different (helical) topology.

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1 A Casimir element C is a member of the *center* of the Poisson algebra, i.e. $[C, G] = 0$ for all G. For finite-dimensional systems, we may regard phase space as the dual of the Lie algebra and Casimir leaves as coadjoint orbits. Unfortunately, in infinite dimensions, there are functional analysis challenges that limit this interpretation (see, e.g., [KHE 58]). Note that a Casimir leaf is not necessarily a symplectic leaf, because the kernel of a Poisson operator may not be fully integrable when the Poisson operator has singularities (see, e.g., [NAR 13, YOS 13a]).

2 The core of the topological constraint is epitomized by Kelvin's circulation law: in a barotropic fluid, the Lie derivative of fluid momentum is an exact differential one-form; thus, the circulation (comoving loop-integral of the fluid momentum, or the surface-integral of the vorticity) is conserved. In a plasma (charged fluid), the momentum combines with the electromagnetic potential (gauge field), and the corresponding *canonical vorticity* is the combination of the fluid vorticity and magnetic field. In the magnetohydrodynamic model, the canonical vorticity of the electron fluid is approximated by the magnetic field, neglecting the electron mass. The circulation law for the magnetic field implies that the magnetic flux on each fluid element is conserved, the so-called Alfvén law.

In the next section, we begin by reviewing some aspects of the basic framework of Hamiltonian mechanics. In section 18.3, we then consider an example of magnetized particles, in order to establish a connection between adiabatic invariants and Casimir elements. In section 18.4, we will formulate a systematic method of *canonization* by adding "angle variables" that result in the unfreezing of Casimir elements. As is now well known, the infinite-dimensional mechanics of a plasma can be formulated as a noncanonical Hamiltonian system on a phase space of Eulerian variables (see, e.g., [MOR 98]). After a short review of the Hamiltonian formalism of MHD and its application to the tearing-mode theory (section 18.5.1), we will formulate a (formal) singular perturbation that gives rise to a tearing-mode instability, and finally discuss its physical implications (section 18.5.2).

18.2. Preliminaries: noncanonical Hamiltonian systems and Casimir invariants

We denote by $\boldsymbol{z} = (q^1, \cdots, q^m, p^1, \cdots, p^m)$ the *state vector*, a point in an affine space $X = \mathbb{R}^{2m}$ (to be called *phase space*)[3]. A canonical Hamiltonian system is endowed with a *Hamiltonian* $H(\boldsymbol{z})$ (a real function on the phase space X) and a $2m \times 2m$ antisymmetric regular matrix

$$J_c := \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix},$$

where I_m and 0_m are the m-dimensional identity and nullity, respectively. (In the following, we will write just I or 0 without specifying the dimension, especially when we consider an infinite-dimensional space.) We call the *canonical Poisson operator* (matrix) J_c. The equations of motion (Hamilton's equations) are written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z} = J_c \partial_{\boldsymbol{z}} H(\boldsymbol{z}), \qquad [18.1]$$

whence an equilibrium point is seen to satisfy $\partial_{\boldsymbol{z}} H(\boldsymbol{z}) = 0$. Defining a Poisson bracket by

$$[a, b] := (\partial_{z_i} a) J_{ij} (\partial_{z_j} b),$$

the rate of change of an observable $f(\boldsymbol{z})$ is determined by

$$\frac{\mathrm{d}}{\mathrm{d}t} f = [f, H].$$

3 Usually, phase space is identified as a cotangent bundle T^*M of a smooth manifold M of dimension m, on which a symplectic two-form $\omega = (1/2) J_{c,k\ell} dz^k \wedge dz^\ell$ (the vorticity of a canonical one-form) defines symplectic geometry.

We may generalize the Poisson operator J to be a function $J(z)$ of an arbitrary dimension $n \times n$ (here we assume a finite n, while we will consider infinite-dimensional systems later). A *noncanonical* Hamiltonian system allows $J(z)$ to be singular, i.e., Rank$J(z)$ may be less than n and can change as a function of z (while the corresponding Poisson bracket must satisfy Jacobi's identity). Then, the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t} z = J(z)\partial_z H(z). \quad [18.2]$$

A *Casimir element* $C(z)$ is a solution to a partial differential equation (PDE)

$$J(z)\partial_z C(z) = 0, \quad [18.3]$$

which implies that $[C,F] = 0$ for every F. Therefore, C is a constant of motion ($dC/dt = [C,H] = 0$ for any Hamiltonian H).

Obviously, if Rank$J(z) = n$ (the dimension of the phase space), [18.3] has only the trivial solution ($C =$ constant). If the dimension ν of Ker($J(z)$) does not change, the solution of (18.3) may be constructed by "integrating" the elements of Ker($J(z)$) – then the Casimir leaves are symplectic manifolds. This expectation turns out to be true provided that the Poisson bracket satisfies the Jacobi identity and $m - \nu$ is an even number (Lie–Darboux theorem). However, the point where the rank of $J(z)$ changes is a singularity of the PDE [18.3] [MOR 98], from which singular Casimir elements are generated [YOS 13a].

When we have a Casimir element $C(z)$ in a noncanonical Hamiltonian system, a transformation of the Hamiltonian $H_(z)$ such as

$$H(Z) \mapsto H_\mu(z) = H(z) - \mu C(z) \quad [18.4]$$

(with an arbitrary real constant μ) does not change the dynamics. In fact, Hamilton's equations [18.2] are invariant under this transformation.

We call the transformed Hamiltonian $H_\mu(z)$ an *energy-Casimir* function [HAZ 84, KRU 58, MOR 98, ARN 98].

Interpreting the parameter μ as a Lagrange multiplier of the equilibrium variational principle, $H_\mu(z)$ is the effective Hamiltonian with the constraint that restricts the Casimir element $C(z)$ to be a given value (since $C(z)$ is a constant of motion, its value is fixed by its initial value). As we will see in some examples, Hamiltonians are rather simple, often being "norms" on the phase space. However, an energy-Casimir functional may have a non-trivial structure. Geometrically, $H_\mu(\boldsymbol{z})$ is the distribution of $H(\boldsymbol{z})$ on a Casimir leaf (hypersurface of $C(\boldsymbol{z}) = \text{constant}$). If Casimir leaves are distorted with respect to the energy norm, the effective Hamiltonian may have a complex distribution on the leaf, which is, in fact, the origin of various interesting structures in noncanonical Hamiltonian systems.

18.3. Foliation by adiabatic invariants

Here, we study an example of noncanonical Hamiltonian mechanics (and the creation of interesting structures on Casimir leaves) in which Casimir elements originate from adiabatic invariants.

The Hamiltonian of a charged particle is the sum of the kinetic energy and the potential energy: $H = mv^2/2 + q\phi$, where $\boldsymbol{v} := (\boldsymbol{P} - q\boldsymbol{A})/m$ is the velocity, $\boldsymbol{P}$ is the canonical momentum, $(\phi, \boldsymbol{A})$ is the electromagnetic four-potential, m is the particle mass and q is the charge. Needless to say, a magnetic field does not change the value of energy, and the standard Boltzmann distribution function is independent of the magnetic field. However, in the vicinity of a dipole magnetic field rooted in a star or planet, for example, we often find a plasma clump with a rather steep density gradient. In such a situation, the so-called *inward diffusion* drives charged particles toward the inner higher density region, which is seemingly opposite to the natural direction of diffusion (normally, diffusion is a process of flattening distributions of physical quantities). Creation of such a macroscopic structure can be explained only by delineating a fundamental difference between a macroscopic hierarchy and basic microscopic mechanics. Since the magnetic field does not cause any change in the energy of particles, there is no way to revise the energy in the calculation of the equilibrium state. Instead, the problem is solved by finding an appropriate "phase space" (or an ensemble) on which the Boltzmann distribution is achieved; the identification of an appropriate macroscopic phase space is nothing but the formulation of what we call a "scale hierarchy".

Magnetized particles live in an effective phase space that is *foliated* by adiabatic invariants associated with periodic motions of particles. Denoting by $\boldsymbol{v}_\parallel$ and $\boldsymbol{v}_\perp$ the parallel and perpendicular (with respect to the local magnetic field) components of the velocity, we may write

$$H = \frac{m}{2}v_\perp^2 + \frac{m}{2}v_\parallel^2 + q\phi. \qquad [18.5]$$

The velocities are related to the mechanical momentum via $\boldsymbol{p} := m\boldsymbol{v}$, $\boldsymbol{p}_\parallel := m\boldsymbol{v}_\parallel$ and $\boldsymbol{p}_\perp := m\boldsymbol{v}_\perp$. In a strong magnetic field, $\boldsymbol{v}_\perp$ can be decomposed into a small-scale cyclotron motion $\boldsymbol{v}_c$ and a macroscopic guiding-center drift motion $\boldsymbol{v}_d$. The periodic cyclotron motion $\boldsymbol{v}_c$ can be "quantized" to write $mv_c^2/2 = \mu\omega_c(\boldsymbol{x})$ in terms of the magnetic moment μ and the cyclotron frequency $\omega_c(\boldsymbol{x})$; the adiabatic invariant μ and the gyration phase $\vartheta_c := \omega_c t$ constitute an action-angle pair. The macroscopic part of the perpendicular kinetic energy is expressed as $mv_d^2/2 = (P_\theta - q\psi)^2/(2mr^2)$, where P_θ is the angular momentum in the θ direction and r is the radius from the geometric axis. In terms of the canonical-variable set $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_\parallel, \theta, P_\theta)$, the Hamiltonian of the guiding center (or, the quasi-particle) becomes

$$H_c = \mu\omega_c + \frac{1}{2m}p_\parallel^2 + \frac{1}{2m}\frac{(P_\theta - q\psi)^2}{r^2} + q\phi. \qquad [18.6]$$

Note that the energy of the cyclotron motion has been quantized in terms of the frequency $\omega_c(\boldsymbol{x})$ and the action μ; the gyro-phase ϑ_c has been coarse grained (integrated to yield 2π).

Now, we formulate the "macroscopic hierarchy" on which charged particles create a thermal equilibrium. The adiabatic invariance of the magnetic moment μ imposes a *topological constraint* on the motion of particles; it is this constraint that is the root cause of macroscopic hierarchy and structure formation. The Poisson operator on the total (microscopic) phase space, spanned by the canonical variables $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_\parallel, \theta, P_\theta)$, is a canonical symplectic matrix

$$J := \begin{pmatrix} J_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}, \quad J_c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad [18.7]$$

The equations of motion for the Hamiltonian H_c are written as $\mathrm{d}z^j/\mathrm{d}t = [z^j, H_c]$. Note that the quantization of the cyclotron motion in H_c suppresses change in μ.

To extract the macroscopic hierarchy, we "separate out" the microscopic variables (ϑ_c, μ) by modifying the Poisson operator as

$$J_{nc} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}. \qquad [18.8]$$

The Poisson bracket $[F,G]_{nc} := \langle \partial_{\boldsymbol{z}} F, J_{nc}\partial_{\boldsymbol{z}} G\rangle$ determines the kinematics on the macroscopic hierarchy. The corresponding kinetic equation $\partial_t f + [H_c, f]_{nc} = 0$ reproduces the familiar drift-kinetic equation. The kernel of J_{nc} makes the Poisson bracket $[\,,\,]_{nc}$ *noncanonical* [MOR 98]. Evidently, μ is a Casimir element (more generally $C = g(\mu)$ with g being any smooth function). The level-set of μ, a leaf of the Casimir foliation, identifies what we may call the *macroscopic hierarchy*.

By applying Liouville's theorem to the Poisson bracket $[\,,\,]_{nc}$, the invariant measure on the macroscopic hierarchy is $\mathrm{d}^4 z = \mathrm{d}^6 z/(2\pi \mathrm{d}\mu)$, the total phase-space measure modulo the microscopic measure. The most probable state (statistical equilibrium) on the macroscopic ensemble maximizes the entropy $S = -\int f \log f \,\mathrm{d}^6 z$ for a given particle number $N = \int f \mathrm{d}^6 z$, a quasi-particle number $M = \int \mu f \mathrm{d}^6 z$ and an energy $E = \int H_c f \mathrm{d}^6 z$. Then, the distribution function is

$$f = f_\alpha := Z^{-1} e^{-(\beta H_c + \alpha\mu)}, \qquad [18.9]$$

where α, β and $\log Z - 1$ are, respectively, the Lagrange multipliers on M, E, and N. In this *grand-canonical distribution function*, α/β is the chemical potential associated with the quasi-particles[4]. The factor $e^{-\alpha\mu}$ in f_α yields a direct ω_c dependence of the coordinate-space density

$$\rho = \int f_\alpha \frac{2\pi\omega_c}{m} \mathrm{d}\mu \mathrm{d}v_d \mathrm{d}v_\parallel \propto \frac{\omega_c(\boldsymbol{x})}{\beta\omega_c(\boldsymbol{x}) + \alpha}, \qquad [18.10]$$

which demonstrates the creation of a density clump near the dipole magnetic field [YOS 13b].

18.4. Canonization atop Casimir leaves

The aim of this section is to formulate a systematic method for the "canonization" of a noncanonical system by embedding the system into a higher dimension phase space; Casimir elements become adiabatic invariants associated with a symmetry (at a macroscale hierarchy) of a Hamiltonian.

18.4.1. *Extension of the phase space and canonization*

Let J be a Poisson matrix on an n-dimensional phase space $X = \mathbb{R}^n$ parameterized by $\boldsymbol{z} = (z_1, \cdots, z_n)$. We assume that $\mathrm{Ker}(J)$ has a dimension ν and $n - \nu$ is an even number. We also assume that $\mathrm{Ker}(J)$ is spanned by Casimir invariants $C_1, \cdots, C_\nu$, i.e.

$$\mathrm{Ker}(J) = \{\nabla C_1, \cdots, \nabla C_\nu\}. \qquad [18.11]$$

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4 We can also derive [18.9] by an *energy-Casimir function*. With a Casimir element μ, we can transform the Hamiltonian as $H_c \mapsto H_\alpha := H_c + \alpha\mu$ (α is an arbitrary constant) without changing the macroscopic dynamic. The Boltzmann distribution with respect to H_α is equivalent to [18.9]. This equivalency was discussed in greater generality in [MOR 87].

Our aim is to find the "minimum" extension of the phase space and a *canonical* Poisson matrix $\tilde{J}$ by which the Casimir invariants are re-interpreted as adiabatic invariants – an appropriate perturbation of the Hamiltonian will then give a near-integrable system in the vicinity of the original Casimir leaves. The target phase space must be of dimension $\tilde{n} := n + \nu$ (even number) consisting of $z_1, \cdots, z_n$ and additional $\vartheta_1, \cdots, \vartheta_\nu$.

Before formulating such a minimum system, we note that we may formally produce a "larger" system; the simplest method of extension and canonization is to double the phase space: let $\boldsymbol{z}_{\times 2} := (z_1, \cdots, z_n, \chi_1, \cdots, \chi_n)$, and

$$J_{\times 2} := \left(\begin{array}{c|c} J & L(\chi) \\ \hline -{}^t L(\chi) & 0 \end{array} \right), \qquad [18.12]$$

where $L(\chi)$ is a certain regular $n \times n$ matrix. To satisfy the Jacobi identity, $L(\chi)$ must satisfy the Maurer–Cartan equation; see equation [292] of [MOR 98].

18.4.2. *"Minimum" canonization invoking Casimir invariants*

It is generally difficult to reduce $J_{\times 2}$ to $\tilde{J}$ of dimension $\tilde{n} \times \tilde{n}$; to separate $2n - \tilde{n}$ variables from $\boldsymbol{z}_{\times 2}$, these variables and the remaining $\tilde{n}$ variables must be independent, implying a "separation of variables."

Our strategy is to use the Casimir foliation [18.11] of the phase space. We first canonize J on $X/\mathrm{Ker}(J)$. Let

$$\boldsymbol{z}' = (\zeta_1, \cdots, \zeta_{n-\nu}, C_1, \cdots, C_\nu) \in \mathbb{R}^n,$$

by which J is transformed into a Darboux standard form

$$J' = \left(\begin{array}{c|c} J_c & \\ \hline & 0_\nu \end{array} \right), \qquad [18.13]$$

We can extend J' to an $\tilde{n} \times \tilde{n}$ canonical matrix such that

$$J_{ex} = \left(\begin{array}{c|cc} J_c & & \\ \hline & 0_\nu & -I_\nu \\ & I_\nu & 0_\nu \end{array} \right). \qquad [18.14]$$

The corresponding variables are denoted by

$$z_{ex} = (\zeta_1, \cdots, \zeta_{n-\nu}, C_1, \cdots, C_\nu, \vartheta_1, \cdots, \vartheta_\nu) \in \mathbb{R}^{\tilde{n}}.$$

An interesting property of this extended, canonized Poisson matrix J_{ex} is that the elements are independent of the additional variables $\boldsymbol{\vartheta}$, which is in marked contrast to the simple extension $J_{\times 2}$ defined in [18.12].

18.5. Application to tearing-mode theory

In this section, we put the method of unfreezing Casimir elements to the test by studying the tearing-mode instability from the perspective of the noncanonical Hamiltonian formalism. The system is of infinite dimension, hence the formulation needs an appropriate functional analytical setting. Here, we invoke a simple incompressible ideal MHD model.

18.5.1. *Helicity and Beltrami equilibria*

18.5.1.1. *MHD system*

Let $\boldsymbol{V}$ and $\boldsymbol{B}$ denote the fluid velocity and magnetic field of a plasma. Here, we consider an incompressible flow, $\nabla \cdot \boldsymbol{V} = 0$, hence both $\boldsymbol{V}$ and $\boldsymbol{B}$ are solenoidal vector fields. The governing equations are (in the so-called Alfvén units)

$$\begin{aligned} &\partial_t \boldsymbol{V} - \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \\ &\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}), \end{aligned} \qquad [18.15]$$

where p denotes the fluid pressure. We consider a three-dimensional bounded domain Ω surrounded by a perfectly conducting boundary $\partial\Omega$; the boundary conditions are (denoting by $\boldsymbol{n}$ the normal trace onto $\partial\Omega$)

$$\boldsymbol{n} \cdot \boldsymbol{V} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{B} = 0, \quad (\text{on } \partial\Omega). \qquad [18.16]$$

The state vector $\boldsymbol{u} = {}^t(\boldsymbol{V}, \boldsymbol{B})$ belongs to the phase space $X = L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$, where

$$L^2_\sigma(\Omega) := \{\boldsymbol{u} \in L^2(\Omega); \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{n} \cdot \boldsymbol{u} = 0\}, \qquad [18.17]$$

which is a closed subspace of $L^2(\Omega)$ (we endow the Hilbert space X with the standard L^2 inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ and the norm $\|\boldsymbol{u}\|$). We denote by $\mathscr{P}_\sigma$ the projector onto $L^2_\sigma(\Omega)$. Defining a Hamiltonian and a Poisson operator by

$$H(\boldsymbol{u}) := \frac{1}{2}\left(\|\boldsymbol{V}\|^2 + \|\boldsymbol{B}\|^2\right), \tag{18.18}$$

$$\mathscr{J}(\boldsymbol{u}) := \begin{pmatrix} -\mathscr{P}_\sigma(\nabla \times \boldsymbol{V})\times & \mathscr{P}_\sigma(\nabla \times \circ) \times \boldsymbol{B} \\ \nabla \times [\circ \times \boldsymbol{B}] & 0 \end{pmatrix}, \tag{18.19}$$

the MHD system [18.15] is cast into the Hamiltonian form

$$\partial_t \boldsymbol{u} = \mathscr{J}(\boldsymbol{u}) \partial_{\boldsymbol{u}} H(\boldsymbol{u}), \tag{18.20}$$

(see [MOR 80, MOR 98, CHA 12]) where $\partial_{\boldsymbol{u}}$ is the gradient (of Lipschitz continuous functionals [CLA 75]) in the Hilbert space X. Here, we define $\mathscr{J}(\boldsymbol{u})$ on a subdomain of C^∞-functions in the phase space X, which suffices to find regular equilibrium points (see [YOS 13a] for more precise definitions).

18.5.1.2. *Beltrami eigenfunctions*

The Poisson operator $\mathscr{J}(\boldsymbol{u})$ has two independent Casimir elements (denoting by $\boldsymbol{A}$ the vector potential of $\boldsymbol{B}$)

$$C_1(\boldsymbol{u}) := \frac{1}{2}\int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}^3x, \quad C_2(\boldsymbol{u}) := \int_\Omega \boldsymbol{V} \cdot \boldsymbol{B}\, \mathrm{d}^3x, \tag{18.21}$$

which, respectively, represent the magnetic helicity and the cross helicity. They impose topological constraints on the field lines [MOF 78]. The "Beltrami equilibrium" is an equilibrium point of the energy-Casimir functional $H(\boldsymbol{u}) - \mu_1 C_1(\boldsymbol{u}) - \mu_2 C_2(\boldsymbol{u})$. Here, we consider a subclass of equilibrium points assuming $\mu_2 = 0$. Then, $\boldsymbol{V} = 0$ (invoking $\mu_2 \neq 0$, we obtain a larger set of equilibria with a finite $\boldsymbol{V}$). The determining equation for $\boldsymbol{B}$ is (denoting $\mu_1 = \mu$)

$$\nabla \times \boldsymbol{B} - \mu \boldsymbol{B} = 0, \tag{18.22}$$

which reads as an eigenvalue problem of the curl operator [YOS 90]. The solution (to be denoted by $\boldsymbol{B}_\mu$) is often called a *Taylor relaxed state* [TAY 74, TAY 86].

Although the Beltrami equation [18.22] and the homogeneous boundary conditions [18.16] are seemingly homogeneous equations, there is a "hidden inhomogeneity" when Ω is multiply connected (then, the boundary conditions

[18.16] are insufficient to determine a unique solution). To delineate the "topological inhomogeneity" of the Beltrami equation, we first make Ω into a simply connected domain Ω_S by inserting cuts Σ_ℓ across each handle of Ω: $\Omega_S := \Omega \setminus (\cup_{\ell=1}^{\nu} \Sigma_\ell)$ (where ν is the *genus* of Ω). The *fluxes* of $\boldsymbol{B}$ are given by (denoting by $\mathrm{d}\boldsymbol{\sigma}$ the surface element on Σ_ℓ) $\Phi_\ell(\boldsymbol{B}) := \int_{\Sigma_\ell} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\sigma}$, which are constants of motion. To separate these fixed degrees of freedom, we invoke the Hodge–Kodaira decomposition $L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_{\mathrm{H}}(\Omega)$, where

$$L^2_\Sigma(\Omega) := \{\boldsymbol{u} \in L^2(\Omega); \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{n} \cdot \boldsymbol{u} = 0, \Phi_\ell(\boldsymbol{u}) = 0 \ (\forall \ell)\}, \qquad [18.23a]$$

$$L^2_{\mathrm{H}}(\Omega) := \{\boldsymbol{u} \in L^2(\Omega); \nabla \times \boldsymbol{u} = 0, \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{n} \cdot \boldsymbol{u} = 0\}. \qquad [18.23b]$$

The dimension of $L^2_{\mathrm{H}}(\Omega)$, the space of *harmonic fields* (or *cohomologies*), is equal to the genus ν of Ω. We decompose the total $\boldsymbol{B} \in L^2_\sigma(\Omega)$ into the fixed harmonic "vacuum" field $\boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)$ (which carries the given fluxes $\Phi_1, \cdots, \Phi_\nu$) and a residual component $\boldsymbol{B}_\Sigma$ driven by currents within the plasma volume Ω, i.e.

$$\boldsymbol{B} = \boldsymbol{B}_\Sigma + \boldsymbol{B}_{\mathrm{H}}, \quad [\boldsymbol{B}_\Sigma := \mathscr{P}_\Sigma \boldsymbol{B} \in L^2_\Sigma(\Omega), \boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)], \qquad [18.24]$$

where $\mathscr{P}_\Sigma$ denotes the orthogonal projector from $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$.

Now, the Beltrami equation [18.22] reads as an inhomogeneous equation (denoting $\nabla \times$ by curl)

$$(\mathrm{curl} - \mu)\boldsymbol{B}_\Sigma = \mu \boldsymbol{B}_{\mathrm{H}}, \qquad [18.25]$$

where the harmonic field $\boldsymbol{B}_{\mathrm{H}}$ is uniquely determined by the fluxes $\Phi_1, \cdots, \Phi_\nu$. When $\boldsymbol{B}_{\mathrm{H}}$ and μ are given, we solve [18.25] for $\boldsymbol{B}_\Sigma$ to obtain the Beltrami magnetic field $\boldsymbol{B}_\mu = \boldsymbol{B}_\Sigma + \boldsymbol{B}_{\mathrm{H}}$. If $\boldsymbol{B}_{\mathrm{H}} = 0$, [18.25] has solutions only for discrete eigenvalues $\mu \in \{\lambda_1, \lambda_2, \cdots\} =: \sigma_p(\mathscr{S})$ of the *self-adjoint curl operator* $\mathscr{S}$ defined on the operator domain [YOS 90]

$$D(\mathscr{S}) = H^1_{\Sigma\Sigma}(\Omega) := \{\boldsymbol{u} \in L^2_\Sigma(\Omega) \cap H^1(\Omega); \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)\}. \qquad [18.26]$$

If $\boldsymbol{B}_H \neq 0$, [18.25] has a non-trivial solution for every $\mu \notin \sigma_p(\mathscr{S})$ [YOS 90]. Moreover, if the vector potential $\boldsymbol{A}_H$ of $\boldsymbol{B}_H$ and the eigenfunction $\boldsymbol{\omega}_j$ of $\mathscr{S}$ belonging to an eigenvalue λ_j are orthogonal (i.e. $\langle \boldsymbol{A}_H, \boldsymbol{\omega}_j \rangle = 0$), the inhomogeneous equation [18.25] has a solution $\boldsymbol{G}_j$ at $\mu = \lambda_j$ even with $\boldsymbol{B}_H \neq 0$. Then, $\mu = \lambda_j$ is a *bifurcation point* of two branches of Beltrami fields, $\boldsymbol{B}_\mu$ with $\mu \gtrsim \lambda_j$ and $\boldsymbol{B}_{\lambda_j,\alpha} = \boldsymbol{G}_j + \alpha \boldsymbol{\omega}_j$ ($\alpha \in \mathbb{R}$), and the latter has a smaller energy for a given helicity C_1 and $\boldsymbol{B}_H$ [YOS 12].

18.5.1.3. *Linearization near the Beltrami equilibrium and tearing mode*

In the neighborhood of a Beltrami equilibrium, we find an infinite number of Casimir elements stemming from the resonant singularity of the Poisson operator, which foliate the phase space and separate the bifurcated Beltrami equilibria on a common helicity leaf.

We linearize the MHD equations. Since the Beltrami equilibrium $\boldsymbol{u}_\mu = {}^t(0, \boldsymbol{B}_\mu)$ is a stationary point of the energy-Casimir functional $H_\mu = H - \mu C_1$, the linearization of Hamilton's equation is rather simple: denoting by $\tilde{\boldsymbol{u}} = {}^t(\tilde{\boldsymbol{V}}, \tilde{\boldsymbol{B}})$ the perturbed state vector, we define linearized Hamiltonian and Poisson operators by

$$\mathscr{H}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \mu \mathscr{S}^{-1} \end{pmatrix}, \qquad [18.27]$$

$$\mathscr{J}_\mu = \begin{pmatrix} 0 & \mathscr{P}_\sigma(\mathrm{curl}\,\circ) \times \boldsymbol{B}_\mu \\ \mathrm{curl}(\circ \times \boldsymbol{B}_\mu) & 0 \end{pmatrix}. \qquad [18.28]$$

Evidently, $\mathscr{H}_\mu$ is a self-adjoint operator for every $\mu \in \mathbb{R}$. The linearized Hamiltonian equation reads

$$\partial_t \tilde{\boldsymbol{u}} = \mathscr{J}_\mu \mathscr{H}_\mu \tilde{\boldsymbol{u}}. \qquad [18.29]$$

In the following, we assume $\mu > 0$. Then, the positive side of the spectrum $\sigma_p(\mathscr{S})$ plays an essential role; for $\mu < 0$, we switch to the negative side of $\sigma_p(\mathscr{S})$. Evidently, $\mu \geq \lambda_1$ destroys the coercivity of $\langle \mathscr{H}_\mu \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle$ with respect to the norm $\|\tilde{\boldsymbol{u}}\|^2$, violating the sufficient condition of stability [YOS 03] (see also [HAZ 84, HOL 85, MOR 86, TAS 92, PET 03]). In fact, a perturbation $\tilde{\boldsymbol{B}} \propto \boldsymbol{\omega}_1$ (the eigenfunction corresponding to λ_1) yields $\langle \mathscr{H}_\mu \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle \leq 0$. However, the negative energy of a perturbation $\tilde{\boldsymbol{B}} \propto \boldsymbol{\omega}_1$ does not necessarily cause an ideal-MHD instability, since motion including $\boldsymbol{\omega}_1$ may be "inhibited" in the Hamiltonian mechanics.

Let us see how Casimir elements foliate the phase space of perturbations: $\mathrm{Ker}(\mathscr{J}_\mu)$ consists of two classes of elements: ${}^t(\boldsymbol{v}, 0)$ and ${}^t(0, \boldsymbol{b})$ with $\boldsymbol{v}$ and $\boldsymbol{b}$ satisfying, respectively,

$$\nabla \times (\boldsymbol{B}_\mu \times \boldsymbol{v}) = 0, \quad \nabla \cdot \boldsymbol{v} = 0, \qquad [18.30a]$$

$$\boldsymbol{B}_\mu \times (\nabla \times \boldsymbol{b}) = 0. \qquad [18.30b]$$

The Casimir elements are, in terms of such $\boldsymbol{v}$ and $\boldsymbol{b}$,

$$C_{\boldsymbol{v}}(\tilde{\boldsymbol{u}}) := \int \tilde{\boldsymbol{V}} \cdot \boldsymbol{v} \, \mathrm{d}^3 x, \quad C_{\boldsymbol{b}}(\tilde{\boldsymbol{u}}) := \int \tilde{\boldsymbol{B}} \cdot \boldsymbol{b} \, \mathrm{d}^3 x. \qquad [18.31]$$

Obviously, we can choose $\boldsymbol{v} = \boldsymbol{B}_\mu$ and $\boldsymbol{b} = \boldsymbol{B}_\mu$. However, far richer solutions arise from the *singularity* of $\mathscr{J}_\mu$.

Here, we concentrate on the "magnetic part" [18.30b], but a similar singular solution $\boldsymbol{v}$ can be constructed for the "flow part" [18.30a]. The determining equation [18.30b] of $\boldsymbol{b}$ can be rewritten as

$$\nabla \times \boldsymbol{b} = \eta \boldsymbol{B}_\mu \qquad [18.32]$$

with some scalar function η. We have already found a solution $\boldsymbol{b} = \boldsymbol{B}_\mu$ and $\eta = \mu$. Here, we seek solutions with a non-constant η. However, η is not a free function; the divergence of both sides of [18.32] yields

$$\boldsymbol{B}_\mu \cdot \nabla \eta = 0, \qquad [18.33]$$

which implies that η is constant along the magnetic field lines. For the integrability of η, the equilibrium field $\boldsymbol{B}_\mu$ must have integrable field lines; a continuous spatial symmetry guarantees this. Here, we consider a *slab geometry*, in which we may write $\boldsymbol{B}_\mu = {}^t\,(0, B_y(x), B_z(x))$. Denoting $\boldsymbol{b} = {}^t\,(0, b_y(x), b_z(x))$, [18.30b] reads as

$$B_y \partial_x b_y + B_z \partial_x b_z = 0, \qquad [18.34]$$

which may be solved for $b_y(x)$, given an arbitrary $b_z(x)$. Furthermore, we have *singular* (hyper function) solutions; let us consider

$$\boldsymbol{b} = {}^t\,(0, b_y(x), b_z(x))\,\mathrm{e}^{\mathrm{i}(k_y y + k_z z)}. \qquad [18.35]$$

Putting $b_y(x) = \mathrm{i}k_y \vartheta(x)$ and $b_z(x) = \mathrm{i}k_z \vartheta(x)$, [18.34] reduces to

$$[B_y(x)k_y + B_z(x)k_z]\partial_x \vartheta(x) = 0, \qquad [18.36]$$

which yields

$$\vartheta(x) = c_0 + c_1 Y(x - x^\dagger), \qquad [18.37]$$

where c_0, c_1 are complex constants, and k_y, k_z and $x^\dagger$ (real constants) are chosen to satisfy the *resonance condition*

$$B_y(x^\dagger)k_y + B_z(x^\dagger)k_z = 0. \qquad [18.38]$$

Then, $\eta = \mathrm{i}(k_y/B_z)\mathrm{e}^{\mathrm{i}(k_y y+k_z z)}\delta(x-x^\dagger)$. From [18.32] we see that this Dirac δ-function solution implies a *current sheet* on the resonant surface $\Gamma^\dagger : x = x^\dagger$. Physically, $\Gamma^\dagger$ represents a thin layer of ideal-MHD plasma that supports a sheet current, which suppresses the change of field-line topology [BOO 11].

In the following, we normalize the kernel element $\boldsymbol{b}$ so that $\|\boldsymbol{b}\|^2 = \langle \boldsymbol{b}, \boldsymbol{b}\rangle = 1$. The singular (hyper function) solution $\boldsymbol{b}$ of [18.35] created by the resonance singularity [18.38], imposes an essential restriction on the range of dynamics; any magnetic perturbation $\tilde{\boldsymbol{B}}$ such that $\langle \tilde{\boldsymbol{B}}, \boldsymbol{b}\rangle \neq 0$ is forbidden to change, because

$$C_{\boldsymbol{b}}(\tilde{\boldsymbol{u}}) = C_{\boldsymbol{b}}(\tilde{\boldsymbol{B}}) := \langle \tilde{\boldsymbol{B}}, \boldsymbol{b}\rangle \qquad [18.39]$$

is an invariant. We call $C_{\boldsymbol{b}}(\tilde{\boldsymbol{B}})$ a "helical-flux Casimir invariant". The equilibrium point of the energy-Casimir functional

$$\mathscr{F}_{\mu,\beta}(\tilde{\boldsymbol{u}}) := \frac{1}{2}\langle \mathscr{H}_\mu \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}\rangle - \beta C_{\boldsymbol{b}}(\tilde{\boldsymbol{u}}) \qquad [18.40]$$

gives the *tearing mode*.

Because of the linearity of the determining equation [18.35], the totality of ${}^t(0,\boldsymbol{b}) \in \mathrm{Ker}(\mathscr{J}_\mu)$ is a linear subspace of the total phase space and it is "integrable", thus foliates the phase space in terms of the Casimir invariants $C_{\boldsymbol{b}}(\tilde{\boldsymbol{u}}) = \langle \tilde{\boldsymbol{B}}, \boldsymbol{b}\rangle$. In the next section, we choose the "dominant helical-flux Casimir" that has the common Fourier coefficients with the helical mode $\boldsymbol{\omega}_1$ of the bifurcated helical Beltrami equilibrium, and define the "minimum extension" that canonizes the corresponding kernel of $\mathscr{J}_\mu$.

18.5.2. *Tearing-mode instability*

18.5.2.1. *Canonization*

Let ${}^t(0,\boldsymbol{b}) \in \mathrm{Ker}(\mathscr{J}_\mu)$. We separate a one-dimensional subspace $\{p\boldsymbol{b};\, p \in \mathbb{R}\}$ from the phase space $L^2_\Sigma(\Omega)$ of magnetic perturbations $\tilde{\boldsymbol{B}}$, and denote by $\wp_\parallel$ the orthogonal projection onto the remaining space

$$\wp_\parallel \tilde{\boldsymbol{B}} := \tilde{\boldsymbol{B}} - \langle \tilde{\boldsymbol{B}}, \boldsymbol{b}\rangle \boldsymbol{b}.$$

We also denote

$$\wp_\perp \tilde{\boldsymbol{B}} := \langle \tilde{\boldsymbol{B}}, \boldsymbol{b}\rangle \boldsymbol{b} = C_{\boldsymbol{b}}(\tilde{\boldsymbol{B}})\boldsymbol{b},$$

and decompose $\tilde{\boldsymbol{B}} = \wp_\parallel \tilde{\boldsymbol{B}} + \wp_\perp \tilde{\boldsymbol{B}}$. Writing the state vector as $\tilde{\boldsymbol{u}}' = {}^t(\tilde{\boldsymbol{V}}, \wp_\parallel \tilde{\boldsymbol{B}}, \wp_\perp \tilde{\boldsymbol{B}})$, and denoting $\mathscr{K}_\mu := (1 - \mu \mathscr{S}^{-1})$, the Hamiltonian and Poisson operators read

$$\mathscr{H}'_\mu = \left(\begin{array}{cc:c} 1 & 0 & \\ 0 & \wp_\parallel \mathscr{K}_\mu & \wp_\parallel \mathscr{K}_\mu \\ \hdashline & \wp_\perp \mathscr{K}_\mu & \wp_\perp \mathscr{K}_\mu \end{array} \right), \qquad [18.41]$$

$$\mathscr{J}'_\mu = \left(\begin{array}{cc:c} 0 & (\mathrm{curl} \wp_\parallel \circ) \times \boldsymbol{B}_\mu & \\ \wp_\parallel \mathrm{curl}(\circ \times \boldsymbol{B}_\mu) & 0 & \\ \hdashline & & 0 \end{array} \right). \qquad [18.42]$$

Note that the kernel ${}^t(0, \boldsymbol{b})$ has been separated from the upper left block of the Poisson operator.

Now, we introduce an adjoint variable q to extend the phase space

$$\tilde{\boldsymbol{u}}_{ex} = {}^t(\tilde{\boldsymbol{V}}, \wp_\parallel \tilde{\boldsymbol{B}}, \wp_\parallel \tilde{\boldsymbol{B}}, q)$$

and define

$$\mathscr{J}_{\mu,ex} = \left(\begin{array}{cc:cc} 0 & (\mathrm{curl} \wp_\parallel \circ) \times \boldsymbol{B}_\mu & & \\ \wp_\parallel \mathrm{curl}(\circ \times \boldsymbol{B}_\mu) & 0 & & \\ \hdashline & & 0 & -1 \\ & & 1 & 0 \end{array} \right), \qquad [18.43]$$

which is "canonized" by extending the variable p. Since the original system does not include q as a variable, we may write

$$\mathscr{H}_{\mu,ex} = \left(\begin{array}{cc:cc} 1 & 0 & & \\ 0 & \wp_\parallel \mathscr{K}_\mu & \wp_\parallel \mathscr{K}_\mu & \\ \hdashline & \wp_\perp \mathscr{K}_\mu & \wp_\perp \mathscr{K}_\mu & 0 \\ & & 0 & 0 \end{array} \right). \qquad [18.44]$$

Evidently, $\wp_\perp \tilde{\boldsymbol{B}} = \langle \tilde{\boldsymbol{B}}, \boldsymbol{b} \rangle \boldsymbol{b}$ is invariant, it was originally a Casimir element, but is now an invariant due to the symmetry of $\mathscr{H}_{\mu,ex}$ with respect to the new variable q.

18.5.2.2. *Singular perturbation*

Perturbing the Hamiltonian with respect to q, we can break the invariance of $p := \langle \tilde{\boldsymbol{B}}, \boldsymbol{b} \rangle = C_{\boldsymbol{b}}(\tilde{\boldsymbol{B}})$. We consider a Hamiltonian

$$\mathscr{H}_{\mu,EX} := \left(\begin{array}{cc:cc} 1 & 0 & & \\ 0 & \wp_{\parallel}\mathscr{K}_\mu & \wp_{\parallel}\mathscr{K}_\mu & \\ \hdashline & \wp_{\perp}\mathscr{K}_\mu & \wp_{\perp}\mathscr{K}_\mu & 0 \\ & & 0 & D \end{array} \right), \qquad [18.45]$$

where D is a parameter that is introduced to couple the original system to the *external variable* q. Note that the original energy $\langle \mathscr{H}_\mu \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} \rangle / 2$ is no longer an invariant; instead, the new total energy $\langle \mathscr{H}_{\mu,EX} \tilde{\boldsymbol{u}}_{ex}, \tilde{\boldsymbol{u}}_{ex} \rangle / 2$ is conserved.

The induced change in the Casimir element (helical flux, which is now denoted by p) is estimated by the canonized block of Hamilton's equations

$$\frac{\mathrm{d}}{\mathrm{d}t} p = -Dq, \quad \frac{\mathrm{d}}{\mathrm{d}t} q = \langle \mathscr{K}_\mu \tilde{\boldsymbol{B}}, \boldsymbol{b} \rangle. \qquad [18.46]$$

For $\tilde{\boldsymbol{B}} = p\boldsymbol{\omega}_1$ (the eigenfunction determining the bifurcated fiducial-energy equilibrium), we may estimate

$$\langle \mathscr{K}_\mu \tilde{\boldsymbol{B}}, \boldsymbol{b} \rangle = \langle (1 - \mu \mathscr{S}^{-1}) \boldsymbol{\omega}_1, \boldsymbol{b} \rangle p = (1 - \mu/\lambda_1) \langle \boldsymbol{\omega}_1, \boldsymbol{b} \rangle p.$$

Absorbing the sign of $\langle \boldsymbol{\omega}_1, \boldsymbol{b} \rangle$ into p, we assume $\gamma := \langle \boldsymbol{\omega}_1, \boldsymbol{b} \rangle > 0$. For simplicity, let us assume that D is a constant number. The system [18.46] has the Hamiltonian

$$H_p := \left(1 - \frac{\mu}{\lambda_1} \right) \gamma \frac{p^2}{2} + D \frac{q^2}{2}. \qquad [18.47]$$

This sub system Hamiltonian describes the coupling of the original (unperturbed) Hamiltonian system with an "external energy" $Dq^2/2$. If this external energy is positive (i.e. $D > 0$), the "internal energy" of the original system may "dissipate" through the coupling. The factor $(1 - \mu/\lambda_1)$ of the "kinetic energy" part of the Hamiltonian H_p may be interpreted as an effective (reciprocal) mass of the tearing mode – beyond the *bifurcation point* $\mu = \lambda_1$, the effective mass becomes negative, and the "negative-energy mode" can grow by absorbing energy from the positive-energy source $Dq^2/2$.

18.6. Conclusion

We have described several facets of noncanonical Hamiltonian systems. Namely, the Poisson operator (field tensor) of a noncanonical Hamiltonian system has a non-trivial kernel (and thus, a cokernel) that foliates the phase space (Poisson manifold), imposing topological constraints on the dynamics. The Hamiltonian (energy) of a weakly coupled macroscopic system (such as a normal fluid or a plasma) is usually rather simple – being a convex functional (typically a quadratic form) by which we can define an energy norm on the phase space. However, an "effective energy" may have a considerably non-trivial distribution on the actual phase space of constrained variables, which is a "distorted" manifold (or a leaf) immersed in the total space. Interesting structures created in a fluid or a plasma may be delineated by unearthing leaves of the phase space and analyzing their distortion with respect to the energy norm. When we can "integrate" the kernel of the Poisson operator to construct Casimir elements, the Casimir leaves foliate the Poisson manifold and, then, the effective energy is the energy-Casimir functional.

In addition, here we have proposed a model for a physical process that removes the constraints of Casimir elements and enables the system to seek lower energy states on different Casimir leaves. By invoking an extended phase space, we canonized the Poisson operator and introduced a coupling of the original ideal system with an external energy source – the exchange of energy between the original system and the connected external system thus described a "dissipation" process. This formulation is based on the method of "minimum canonization" that interprets Casimir elements as "adiabatic invariants," and "unfreezes" the Casimir elements to be dynamic, by perturbing the Hamiltonian with respect to the new angle variable added to the phase space; such perturbations increase the number of degrees of freedom and are therefore a kind of *singular perturbation*.

The theory was applied to the tearing-mode instability, where a tearing mode was regarded as an equilibrium point on a helical-flux Casimir leaf. As long as the helical flux is constrained, the tearing mode cannot grow. However, it was shown that a singular perturbation that allows the system to change the helical flux can cause a tearing mode to grow if it has an excess energy with respect to a fiducial energy of the Beltrami equilibrium at the bifurcation point.

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18.8. Bibliography

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