Chapter 17

Spectral Stability of Nonlinear Waves in KdV-Type Evolution Equations

This chapter focuses on the spectral stability of nonlinear waves in Korteweg-de Vries (KdV) type evolution equations. The relevant eigenvalue problem is defined by the composition of an unbounded self-adjoint operator with a finite number of negative eigenvalues and an unbounded non-invertible operator ∂_x. The instability index theorem is proven under a generic assumption on the self-adjoint operator both in the case of solitary waves and periodic waves. This result is reviewed in the context of recent results on spectral stability of nonlinear waves in KdV-type evolution equations.

17.1. Introduction

KdV-type evolution equations are defined by the following nonlinear partial differential equation in $(1+1)$ variables:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} E'(u), \quad u(t) \in \mathscr{X}, \qquad [17.1]$$

where $E : \mathscr{X} \to \mathbb{R}$ is a C^2 functional on a subspace $\mathscr{X}$ of Hilbert space L^2 associated with the inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$. A critical point $\phi \in \mathscr{X}$ of the Hamiltonian functional E, defined by $E'(\phi) = 0$, represents a nonlinear wave of the KdV-type evolution equation. Depending on the phase space $\mathscr{X}$, ϕ can be a solitary

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wave on an infinite line $\mathbb{R}$ or a $(2L)$-periodic wave on the fundamental period $[-L,L]$. In the following, before section 17.4, we consider solitary waves on an infinite line defined in an L^2-based Sobolev space.

The spectral stability of ϕ is determined by the spectrum of the non-self-adjoint eigenvalue problem:

$$\partial_x E''(\phi)v = \lambda v, \qquad [17.2]$$

where $\mathscr{L} := E''(\phi)$ is a self-adjoint real-valued operator with a dense domain $D(\mathscr{L})$ in $L^2(\mathbb{R})$. Since $\mathscr{L}$ is real-valued, for every eigenvalue $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \neq 0$, there is an eigenvalue $\bar{\lambda} \in \mathbb{C}$. We also assume the Hamiltonian symmetry, that is for every eigenvalue $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \neq 0$, there is an eigenvalue $-\lambda \in \mathbb{C}$. For instance, if $E''(\phi)$ is invariant under the parity transformation, then the Hamiltonian symmetry holds and if $v(x)$ is the eigenvector of the spectral problem [17.2] for λ, then $v(-x)$ is the eigenvector of the spectral problem [17.2] for $-\lambda$.

The nonlinear wave ϕ is *spectrally stable* if $\sigma(\partial_x\mathscr{L}) \subset i\mathbb{R}$ and it is *spectrally unstable* if there is $\lambda_0 \in \sigma(\partial_x\mathscr{L})$ such that $\mathrm{Re}(\lambda_0) > 0$, where $\sigma(\partial_x\mathscr{L})$ denotes the spectrum of the non-self-adjoint eigenvalue problem [17.2]. The corresponding eigenvector v for an eigenvalue $\lambda \in \sigma(\partial_x\mathscr{L})$ belongs to the function space $D(\partial_x\mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R}) \subset L^2(\mathbb{R})$, where $\dot{H}^{-1}(\mathbb{R})$ is the space of all distributions with square integrable antiderivatives. In other words, if $v \in \dot{H}^{-1}(\mathbb{R})$, then $\partial_x^{-1} v \in L^2(\mathbb{R})$.

We assume that the unbounded self-adjoint operator $\mathscr{L}$ from $D(\mathscr{L}) \subset L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ is given by the sum of two operators $\mathscr{L}_0$ and $K_{\mathscr{L}}$, where $\mathscr{L}_0$ is a strongly elliptic unbounded operator with constant coefficients and $K_{\mathscr{L}}$ is a relatively compact perturbation of $\mathscr{L}_0$. Using the Fourier transform $\mathscr{F}$ on $L^2(\mathbb{R})$, we define the image of $\mathscr{L}_0$ as follows:

$$\mathscr{F}(\mathscr{L}_0 u)(k) = \hat{\mathscr{L}}_0(k)\mathscr{F}(u)(k), \quad k \in \mathbb{R}.$$

Since $\mathscr{L}_0$ is unbounded, a coercivity condition holds to yield $\hat{\mathscr{L}}_0(k) \to \infty$ as $|k| \to \infty$. We will further assume the following generic assumptions.

H1). There is $c_0 > 0$ such that $\hat{\mathscr{L}}_0(k) \geq c_0$ for all $k \in \mathbb{R}$. By Weyl's theorem, this implies that the essential spectrum of $\mathscr{L}$ (denoted as $\sigma_e(\mathscr{L})$) is bounded away from zero by a positive number.

H2). The discrete spectrum of $\mathscr{L}$ (denoted as $\sigma_d(\mathscr{L})$) includes a finite number $n(\mathscr{L})$ of negative eigenvalues with eigenvectors in $D(\mathscr{L})$.

H3). $\mathrm{Ker}(\mathscr{L}) = \mathrm{span}\{f_0\}$ with $f_0 \in D(\mathscr{L}) \cap D(\partial_x \mathscr{L} \partial_x) \cap \dot{H}^{-1}(\mathbb{R})$ so that $\phi_0 = \partial_x^{-1} f_0 \in L^2(\mathbb{R})$[1].

H4). $\langle \mathscr{L}^{-1}\phi_0, \phi_0 \rangle \neq 0$. The value of $\langle \mathscr{L}^{-1}\phi_0, \phi_0 \rangle$ is finite because $\langle f_0, \phi_0 \rangle = \langle \partial_x \phi_0, \phi_0 \rangle = 0$.

Under these generic assumptions, we obtain the instability index count, which is analogous to instability index count for nonlinear Schrödinger (NLS) type evolution equations [CHU 10, CUC 05, KAP 04, PEL 05] (see also Chapter 4 in [PEL 11]). To formulate the theorem, let us define the following numbers for the eigenvalue problem [17.2] with the account of algebraic multiplicity of eigenvalues:

– N_r is the number of real positive eigenvalues λ.

– N_c is the number of complex eigenvalues λ in the first open quadrant of $\mathbb{C}$.

– N_i^- is the total negative Krein index[2] associated with the number of imaginary (possibly, embedded) eigenvalues λ with $\mathrm{Im}(\lambda) > 0$.

Our main result is the following theorem.

THEOREM 17.1.– Assume H1–H4. Then:

$$N_r + 2N_c + 2N_i^- = n(\mathscr{L}) - n_0, \qquad [17.3]$$

where $n_0 = 1$ if $\langle \mathscr{L}^{-1}\phi_0, \phi_0 \rangle < 0$ and $n_0 = 0$ if $\langle \mathscr{L}^{-1}\phi_0, \phi_0 \rangle > 0$.

Section 17.2 contains historical notes devoted to theorem 17.1 and other relevant recent results. The proof of theorem 17.1 is developed in section 17.3. A generalization of theorem 17.1 for a periodic nonlinear wave ϕ is given in section 17.4. Section 17.5 discusses possible further developments in the area.

17.2. Historical remarks and examples

The Hamiltonian functional $E(u)$ conserves in time t in the KdV-type evolution equation [17.1]. For many KdV-type evolution equations, another conserved C^2

1 The restrictive assumption $f_0 \in D(\partial_x \mathscr{L} \partial_x)$ is needed because f_0 later defines the projection operator P in the generalized eigenvalue problem [17.15].

2 The negative Krein index of an invariant subspace $E_\lambda \subset L^2$ of the spectral stability problem [17.2] associated with an eigenvalue $\lambda \in i\mathbb{R}$ is the number of non-positive eigenvalues of $\langle \mathscr{L}|_{E_\lambda} u, u \rangle$. These eigenvalues of $\langle \mathscr{L}|_{E_\lambda} u, u \rangle$ cannot be zero if λ is an isolated eigenvalue but may include zero eigenvalue if λ is an embedded eigenvalue.

functional $P(u)$ typically exists, called the momentum functional. For example, for the general fifth-order KdV equation [CHA 97, CRA 94]:

$$u_t = a_1 u_x - a_2 u_{xxx} + a_3 u_{xxxxx} + 3b_1 u u_x - b_2 (u u_{xxx} + 2u_x u_{xx}) + 6b_3 u^2 u_x, \qquad [17.4]$$

where $(a_1, a_2, a_3, b_1, b_2, b_3)$ are real, the energy functional $E(u)$ is well defined in $H^2(\mathbb{R})$:

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} \left(a_1 u^2 + a_2 u_x^2 + a_3 u_{xx}^2 + b_1 u^3 + b_2 u u_x^2 + b_3 u^4 \right) dx, \qquad [17.5]$$

whereas the momentum functional is $P(u) = \|u\|^2$. Without loss of generality, we assume that the phase speed for linear waves in the fifth-order KdV equation [17.4] is non-negative. Using the Fourier transform $\mathscr{F}$, we express this assumption as follows:

$$c_{\text{wave}}(k) = a_1 + a_2 k^2 + a_3 k^4 \geq 0, \quad k \in \mathbb{R}. \qquad [17.6]$$

Assumption [17.6] is needed for assumption H1 in theorem 17.1.

Besides the translational parameter x_0 in $\phi(x - x_0)$, the nonlinear wave ϕ typically has another free parameter c for the constant speed. Accounting speed c, the nonlinear wave is a critical point of the *extended* energy functional $E_c(u) := E(u) + cP(u)$. The second variation of E_c defines the self-adjoint operator $\mathscr{L}_c := E''(\phi) + cP''(\phi)$.

General stability-instability results for the critical points of $E_c(u)$ were obtained in [BON 87, GRI 87, SOU 90], based on the assumption that the self-adjoint linearized operator $\mathscr{L}_c$ has exactly one negative eigenvalue and a simple zero eigenvalue. Using a different method involving modulation equations, Lyapunov stability of positive traveling waves ϕ was also proved by Weinstein [WEI 87]. In the two consequent and influential papers, Pego and Weinstein developed the Evans function analysis of spectral stability [PEG 92] and analysis of asymptotic stability in exponentially weighted spaces [PEG 94] in the context of a generalized KdV equation.

These general results correspond to the case $n(\mathscr{L}) = 1$ in theorem 17.1 (see also [ANG 09] for a recollection of these and many other results). More recently, questions have been raised about the spectral stability of KdV-type nonlinear waves in the cases where $n(\mathscr{L}) > 1$, which are known for equations of the integrable KdV hierarchy [MAD 93, KOD 05]. The result of theorem 17.1 was already claimed as early as 2006 in the context of the fifth-order KdV equation [17.4] [CHU 06], although the final version of this paper was published without the example of the fifth-order KdV equation [CHU 10]. Since that time, a weaker result was obtained by Lin [LIN 08] and a nearly identical result was outlined very recently by Kapitula and Stefanov [KAP 13]. Periodic waves of KdV-type nonlinear evolution equations were treated in [ANG 08, ANG 11, BRO 12a, DEC 10, HAR 08], where results similar to theorem 17.1 were obtained. Therefore, it is fair to restore the original proof of theorem 17.1 following the lines of [CHU 06] and to show how the instability index count for both solitary and periodic waves can naturally be adopted from a general theory in Pontryagin's space [PON 44]. This task is achieved in this chapter with the main goal of showing the universality and simplicity of the proof of the instability index counts by using the generalized eigenvalue problem (that is the linear operator pencil in the terminology of the recent review [KOL 12]).

In the context of the general fifth-order KdV equation [17.4], specific studies of Lyapunov stability of traveling solitary waves were reported in [DIA 99, ILL 92] with the energy-momentum methods. In particular, since the solitary wave satisfies the fourth-order differential equation:

$$a_3\phi'''' - a_2\phi'' + (a_1+c)\phi + \frac{3}{2}b_1\phi^2 - \frac{1}{2}b_2\left(2\phi\phi'' + (\phi')^2\right) + 2b_3\phi^3 = 0, \qquad [17.7]$$

we can verify by direct computations that $\mathscr{L}_c\phi' = 0$ and $\mathscr{L}_c\partial_c\phi = -\phi$, where the prime denotes differentiation in x and ∂_c denotes differentiation in c, whereas:

$$\mathscr{L}_c := a_3\frac{d^4}{dx^4} - a_2\frac{d^2}{dx^2} + a_1 + c + 3b_1\phi(x) - b_2\frac{d}{dx}\phi(x)\frac{d}{dx} - b_2\phi''(x) + 6b_3\phi^2(x). [17.8]$$

Assuming existence and uniqueness (up to translational invariance) of a solitary wave $\phi \in H^2(\mathbb{R})$ with the exponential decay at infinity for $c > 0$ (see [CHA 97, KIC 97, LEV 99] for existence results), we realize that the operator $\mathscr{L}_c$ satisfies assumptions H1–H4 of theorem 17.1 with $\hat{\mathscr{L}}_0(k) = c + c_{\text{wave}}(k) \geq c > 0$, $\phi_0 = \phi$, and:

$$\langle \mathscr{L}_c^{-1}\phi, \phi\rangle = -\langle \partial_c\phi, \phi\rangle = -\frac{1}{2}\frac{d}{dc}\|\phi\|^2.$$

If $n(\mathscr{L}_c) = 1$, the result of theorem 17.1 gives stability of a solitary wave if $\frac{d}{dc}\|\phi\|^2 > 0$ and instability if $\frac{d}{dc}\|\phi\|^2 < 0$, which coincides with the results of the orbital stability theory [ANG 09, BON 87, WEI 87].

The spectral stability of one-humped solitary waves in the fifth-order KdV equation was studied numerically in [BRI 02b], with the use of the symplectic Evans matrix [BRI 02a]. Because $n(\mathscr{L}_c) = 1$ and $\frac{d}{dc}\|\phi\|^2 > 0$ were found, the one-humped solitary waves were shown to be spectrally stable.

One-humped and two-humped solitary waves in the fifth-order KdV equation were numerically approximated in [CHU 07] with a spectral method. Numerical results on eigenvalues of the spectral problem [17.2] were found in full correspondence with the result of theorem 17.1. In particular, two-humped solutions have either $n(\mathscr{L}_c) = 2$ or $n(\mathscr{L}_c) = 3$, depending on whether the individual solitary waves form a bound state at the non-degenerate minimum or maximum points of the effective interaction potential. Since $\frac{d}{dc}\|\phi\|^2 > 0$ for all these solitary waves, the two-humped solutions with $n(\mathscr{L}_c) = 2$ are unstable with $N_r = 1$. Nevertheless, the two-humped solutions with $n(\mathscr{L}_c) = 3$ are spectrally stable, because the single pair of embedded eigenvalues with negative Krein signature $N_i^- = 1$ is structurally stable with respect to parameter continuations [CHU 07]. Similar results were also observed numerically with the computations of the Maslov index for solitary waves in the fifth-order KdV equation [CHA 09].

To finish these remarks, we also mention a similar instability index count obtained for dark solitons in the defocusing NLS equation with an external potential [PEL 08]. Although the symplectic operator for the NLS equation is invertible, the spectral stability problem for dark solitons (solitary waves with non-zero boundary conditions) is defined in terms of a linear self-adjoint operator, where the positive essential spectrum touches zero. Nevertheless, the theory from [CHU 10] was successfully applied to the count of unstable eigenvalues for a dark soliton in a spatially localized potential and illustrated with a number of prototypical examples in [PEL 08]. In this context, a dark soliton persists in a small localized potential if it is located at the non-degenerate minimum or maximum points of the effective potential and is spectrally unstable in both cases. At the maximum point, the dark soliton is unstable with one real eigenvalue $N_r = 1$, whereas at the minimum point, it is unstable with two complex eigenvalues $N_c = 1$. The embedded imaginary eigenvalues with negative Krein signature are structurally unstable with respect to parameter continuations in the defocusing NLS equations and bifurcate into complex unstable eigenvalues (see [PEL 08] for precise computations of these instabilities by using the Evans function for dark solitons).

17.3. Proof of theorem 17.1

We consider the spectral problem [17.2], where the self-adjoint operator $\mathscr{L} = E''(\phi)$ satisfies the assumptions H1–H4 of theorem 17.1.

The proof [CHU 10, CUC 05, KAP 04, PEL 05] of the standard instability index count is not applicable to the KdV-type evolution equations because the symplectic operator ∂_x is not invertible. Nevertheless, the range of the self-adjoint operator $\mathscr{L}$ is defined in $L^2(\mathbb{R})$; hence, bootstrapping arguments imply that the eigenvector v of the spectral problem [17.2] with $\lambda \neq 0$ belongs to the function space $D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R}) \subset L^2(\mathbb{R})$. Therefore, we can define $w = \partial_x^{-1} v \in L^2(\mathbb{R})$ and formally extend the spectral problem [17.2] to the system of two coupled equations:

$$\mathscr{M} w = -\lambda v, \quad \mathscr{L} v = \lambda w, \qquad [17.9]$$

where $\mathscr{M} := -\partial_x \mathscr{L} \partial_x$, $v \in D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R}) \subset L^2(\mathbb{R})$, and $w \in D(\partial_x \mathscr{L} \partial_x) \subset L^2(\mathbb{R})$. The coupled system [17.9] is equivalent to the squared eigenvalue problem $\partial_x \mathscr{L} \partial_x \mathscr{L} v = \lambda^2 v$. We show now that if the coupled system [17.9] has an eigenvalue $\lambda_0 \neq 0$, then it has another eigenvalue $-\lambda_0$ and these two eigenvalues are equivalent to the pair of eigenvalues λ_0 and $-\lambda_0$ of the spectral problem [17.2]. For simplicity of presentation, we only consider the case of simple non-zero eigenvalues in this chapter.

PROPOSITION 17.1.– The coupled system [17.9] has a pair of simple eigenvalues $\pm\lambda_0 \neq 0$ with the eigenvectors $(v_0, \pm w_0) \in D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R}) \times D(\partial_x \mathscr{L} \partial_x)$ if and only if the spectral problem [17.2] has a pair of simple eigenvalues $\pm\lambda_0$ with the eigenvectors $v_\pm = v_0 \pm \partial_x w_0 \in D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R})$.

PROOF.– By the symmetry, if $\lambda_0 \neq 0$ is a simple eigenvalue of the coupled system [17.9] with the eigenvector $(v_0, w_0) \in D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R}) \times D(\partial_x \mathscr{L} \partial_x)$, then $-\lambda_0$ is also a simple eigenvalue of the coupled system [17.9] with the eigenvector $(v_0, -w_0)$. Moreover, v_0 and w_0 are linearly independent.

We differentiate the second equation of the coupled system [17.9] for the eigenvalue λ_0 and add or subtract the first equation of the system to obtain:

$$\partial_x \mathscr{L}(v_0 \pm \partial_x w_0) = \pm\lambda_0 (v_0 \pm \partial_x w_0).$$

Therefore, $v_0 \pm \partial_x w_0 \in \mathrm{Ker}(\partial_x \mathscr{L} \mp \lambda_0)$. By the Hamiltonian symmetry, if $\lambda_0 \in \sigma(\partial_x \mathscr{L})$, then $-\lambda_0 \in \sigma(\partial_x \mathscr{L})$, whereas the algebraic multiplicity of eigenvalues in $\sigma(\partial_x \mathscr{L} \partial_x \mathscr{L})$ equals the algebraic multiplicity of eigenvalues in the coupled system [17.9]. This guarantees that the two eigenvectors $(v_0, \pm w_0)$ of the coupled system [17.9] generate two linearly independent eigenvectors $v_\pm = v_0 \pm \partial_x w_0$ for $\pm\lambda_0 \in \sigma(\partial_x \mathscr{L})$.

To check the converse statement, we assume that $v_\pm$ are linearly independent eigenvectors of $\partial_x \mathscr{L}$ for the eigenvalues $\pm\lambda_0$ in $D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R})$. Then, we define non-zero functions:

$$v_0 := \frac{1}{2}(v_+ + v_-), \quad w_0 := \frac{1}{2}(\partial_x^{-1} v_+ - \partial_x^{-1} v_-), \qquad [17.10]$$

and obtain:

$$\partial_x \mathscr{L} v_0 = \frac{1}{2}\partial_x \mathscr{L}(v_+ + v_-) = \frac{1}{2}\lambda_0(v_+ - v_-) = \lambda_0 \partial_x w_0,$$

so that the integration gives $\mathscr{L} v_0 = \lambda_0 w_0$, that is the second equation of the coupled system [17.9]. Similarly, we check the first equation of the coupled system [17.9] with $\mathscr{M} w_0 = -\lambda_0 v_0$. Therefore, the two eigenvectors $v_\pm$ for $\pm\lambda_0 \in \sigma(\partial_x \mathscr{L})$ generate two linearly independent eigenvectors $(v_0, \pm w_0)$ of the coupled system [17.9] for a pair of eigenvalues $\pm\lambda_0 \neq 0$.

REMARK 17.1.– In many KdV-type evolution equations including the fifth-order KdV equation [17.4], the Hamiltonian symmetry of the spectral problem [17.2] follows from the parity transformation of the eigenvectors, if the nonlinear wave ϕ is symmetric with respect to x. Therefore, if $v_+(x)$ is a solution of $\partial_x \mathscr{L} v_+ = \lambda_0 v_+$, then $v_-(x) := v_+(-x)$ is a solution of $\partial_x \mathscr{L} v_- = -\lambda_0 v_-$. Under this transformation, the components v_0 and w_0 of the coupled system [17.9] for a simple eigenvalue λ_0 are either even or odd functions with respect to x, whereas $v_\pm$ are neither even nor odd.

To study the spectrum of the coupled system [17.9], we shall first understand the spectrum of operator $\mathscr{M}$. Recall again that σ_e and σ_d denote the essential and discrete spectra. From assumptions H1–H4, we obtain the following properties of operator $\mathscr{M}$.

LEMMA 17.1.– Under assumptions H1–H4 on $\mathscr{L}$, operator $\mathscr{M}$ can be extended to a self-adjoint operator with a dense domain $D(\mathscr{M})$ in $L^2(\mathbb{R})$ satisfying the following properties:

H1′). $\sigma_e(\mathscr{M}) \geq 0$.

H2′). $\sigma_d(\mathscr{M})$ includes $n(\mathscr{L})$ negative eigenvalues with eigenvectors in $D(\mathscr{M})$.

H3′). $\mathrm{Ker}(\mathscr{M}) = \mathrm{span}\{\phi_0\}$.

H4′). $\langle \mathscr{M}^{-1} f_0, f_0 \rangle$ is finite and non-zero.

PROOF.– From the decomposition $\mathscr{L} = \mathscr{L}_0 + K_{\mathscr{L}}$, we have the decomposition $\mathscr{M} = \mathscr{M}_0 + K_{\mathscr{M}}$, where $K_{\mathscr{M}} = -\partial_x K_{\mathscr{L}} \partial_x$ is a relatively compact perturbation of $\mathscr{M}_0 = -\partial_x \mathscr{L}_0 \partial_x$. Since $\mathscr{M}_0$ is a linear operator with constant coefficients, we use the Fourier transform $\mathscr{F}$ on $L^2(\mathbb{R})$ to find the image of $\mathscr{M}_0$ as follows:

$$\hat{\mathscr{M}}_0(k) = k^2 \hat{\mathscr{L}}_0(k) \geq 0 \quad \text{for all } k \in \mathbb{R}.$$

Since $\hat{\mathscr{L}}_0(k) \geq c_0$ by assumption H1, we have $\hat{\mathscr{M}}_0(k) \geq 0$ for all $k \in \mathbb{R}$. By Weyl's theorem, this implies that $\sigma_e(\mathscr{M})$ is non-negative, i.e. H1′ holds.

Since $\text{Ker}(\partial_x) = \text{span}\{0\}$ in $L^2(\mathbb{R})$, H3$'$ follows from H3 by direct computations:

$$-\partial_x \mathscr{L} \partial_x f = 0 \;\Rightarrow\; \mathscr{L} \partial_x f = 0 \;\Rightarrow\; \partial_x f \in \text{span}\{\partial_x \phi_0\}, \;\Rightarrow\; f \in \text{span}\{\phi_0\}.$$

Furthermore, $\langle \mathscr{M}^{-1} f_0, f_0 \rangle$ exists because ∂_x^{-1} is well defined in $L^2(\mathbb{R})$ on functions in $L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$. As a result, H4$'$ follows from H4 by means of integration by parts:

$$\langle \mathscr{M}^{-1} f_0, f_0 \rangle = \langle \mathscr{M}^{-1} \partial_x \phi_0, \partial_x \phi_0 \rangle = -\langle \partial_x \mathscr{M}^{-1} \partial_x \phi_0, \phi_0 \rangle = \langle \mathscr{L}^{-1} \phi_0, \phi_0 \rangle.$$

H2$'$ remains to be proven. The negative eigenvalues of $\sigma_d(\mathscr{M})$ are defined from the eigenvalue problem $\mathscr{M} f = \lambda f$, which is rewritten in the following form:

$$-\partial_x \mathscr{L} \partial_x f = \lambda f, \quad f \in D(\partial_x \mathscr{L} \partial_x) \cap \dot{H}^{-1}(\mathbb{R}) \subset L^2(\mathbb{R}). \qquad [17.11]$$

Since $f \in \dot{H}^{-1}(\mathbb{R})$, there exists $g = \partial_x^{-1} f \in L^2(\mathbb{R})$ such that the spectral problem [17.11] can be written in the equivalent form:

$$-\mathscr{L} \partial_x^2 g = \lambda g, \quad g \in D(\mathscr{L} \partial_x^2) \subset L^2(\mathbb{R}). \qquad [17.12]$$

For any $\varepsilon > 0$, the positive operator $(\varepsilon - \partial_x^2)$ is invertible and the inverse operator is defined by the integral representation:

$$(\varepsilon - \partial_x^2)^{-1} f(x) = \frac{1}{2\varepsilon^{1/2}} \int_{\mathbb{R}} e^{-\varepsilon^{1/2}|x-y|} f(y) dy, \quad f \in L^2(\mathbb{R}). \qquad [17.13]$$

Using this integral representation, we define a smoothened version of the eigenvalue problem [17.12] for $h = (\varepsilon - \partial_x^2) g$:

$$\mathscr{L} h = \lambda (\varepsilon - \partial_x^2)^{-1} h, \quad h \in D(\mathscr{L}) \subset L^2(\mathbb{R}). \qquad [17.14]$$

Since $(\varepsilon - \partial_x^2)^{-1}$ is a positive bounded self-adjoint operator for any $\varepsilon > 0$, Sylvester's Law of Inertia (theorem 4.2 in [PEL 11]) applies and the number of negative and zero eigenvalues of the spectral problem [17.14] corresponds to the number of negative and zero eigenvalues of the operator $\mathscr{L}$. By assumptions H2 and H3, there are exactly $n(\mathscr{L})$ negative eigenvalues of the eigenvalue problem [17.14] and a simple zero eigenvalue for any $\varepsilon > 0$.

Because the integral representation [17.13] diverges as $\varepsilon \downarrow 0$ and the operator $\mathscr{L}$ is bounded from below, the negative eigenvalues of the spectral problem [17.14] for

$\varepsilon > 0$ are bounded from below but may *a priori* approach to zero as $\varepsilon \downarrow 0$. However, since the kernel of $(\varepsilon - \partial_x^2)\mathcal{L}$ is simple for any $\varepsilon \geq 0$, the negative eigenvalues are bounded away from zero as $\varepsilon \downarrow 0$. As a result, the spectral problem [17.12] also has $n(\mathcal{L})$ negative eigenvalues, that is H2′ holds[3].

Now we convert the coupled system [17.9] for a pair of simple eigenvalues $\pm\lambda \neq 0$ to a generalized eigenvalue problem for a double eigenvalue. By assumptions H1 and H3, the zero eigenvalue of $\mathcal{L}$ is bounded away from the essential spectrum of $\mathcal{L}$. Let P be the orthogonal projection from $L^2(\mathbb{R})$ to $[\text{span}\{f_0\}]^\perp \subset L^2(\mathbb{R})$. The following result establishes this equivalence.

PROPOSITION 17.2.– The coupled system [17.9] has a pair of simple eigenvalues $\pm\lambda \neq 0$ with eigenvectors $(v, \pm w)$ if and only if the generalized eigenvalue problem:

$$Aw = \gamma K w, \quad w \in \mathcal{H} := D(\mathcal{M}) \cap [\text{span}\{f_0\}]^\perp \subset L^2(\mathbb{R}), \qquad [17.15]$$

where $A := P\mathcal{M}P$ and $K := P\mathcal{L}^{-1}P$ has a double eigenvalue $\gamma = -\lambda^2 \neq 0$ with linearly independent eigenvectors $\partial_x^{-1}v$ and w.

PROOF.– Let $\pm\lambda \neq 0$ be a pair of simple eigenvalues of the coupled system [17.9] with the eigenvectors $(v, \pm w) \in D(\partial_x\mathcal{L}) \cap \dot{H}^{-1}(\mathbb{R}) \times D(\partial_x\mathcal{L}\partial_x)$. Because $\lambda \neq 0$, we have $w = Pw$, that is w is in the range of $\mathcal{L}$. As a result, the second equation of the coupled system [17.9] can be written in the equivalent form:

$$v = \lambda P\mathcal{L}^{-1}Pw + v_0, \quad v_0 \in \text{Ker}(\mathcal{L}). \qquad [17.16]$$

Substituting v into the first equation of the coupled system [17.9] and using the projection operator P again, we obtain a closed equation for w:

$$P\mathcal{M}Pw = -\lambda^2 P\mathcal{L}^{-1}Pw, \quad w \in D(\mathcal{M}) \cap [\text{span}\{f_0\}]^\perp \subset L^2(\mathbb{R}) \qquad [17.17]$$

and a unique expression for v_0:

$$v_0 = -\frac{1}{\lambda}(I - P)\mathcal{M}Pw, \qquad [17.18]$$

3 Another smoothened version of the same eigenvalue problem [17.12] is $(\varepsilon - \partial_x^2)g = \lambda\mathcal{L}^{-1}g$ for all $g \in H^2(\mathbb{R}) \cap [\text{span}\{f_0\}]^\perp$. By the same Sylvester Law of Inertia, there are exactly $n(\mathcal{L})$ negative eigenvalues of this eigenvalue problem for any $\varepsilon > 0$ and the bootstrapping arguments give $g \in D(\mathcal{L}\partial_x^2) \subset L^2(\mathbb{R})$ for the corresponding eigenvectors.

where $\lambda \neq 0$ and $(I - P)$ is the orthogonal projection from $L^2(\mathbb{R})$ to $\text{Ker}(\mathscr{L})$. Therefore, it follows from equation [17.17] that $\gamma = -\lambda^2$ is an eigenvalue of the generalized eigenvalue problem [17.15] with an eigenvector w.

To show that this γ is a double eigenvalue of the generalized eigenvalue problem [17.15], we note that the coupled system [17.9] is invariant with respect to the transformation:

$$\partial_x w \to v \quad \text{and} \quad \partial_x^{-1} v \to w. \qquad [17.19]$$

Therefore, $\partial_x^{-1} v$ is another eigenvector of the generalized eigenvalue problem [17.15] for the same γ. By proposition 17.1, see the equivalence formula [17.10], $\partial_x^{-1} v$ and w are linearly independent.

In the opposite direction, if $\gamma \neq 0$ is a double eigenvalue of the generalized eigenvalue problem [17.15] with linearly independent eigenvectors w_1 and w_2, then for each eigenvector w_1 or w_2, we define v_0 by [17.18] and v by [17.16], which yields linearly independent components v_1 and v_2 of the coupled system [17.9] for the eigenvalue $\lambda = (-\gamma)^{1/2}$. The eigenvalue $\lambda = (-\gamma)^{1/2}$ must be simple (or the multiplicity of the eigenvalue γ in the generalized eigenvalue problem [17.15] exceeds two); therefore, the transformation [17.19] yields the correspondence $v_1 = \partial_x w_2$ and $v_2 = \partial_x w_1$. In other words, only (v_1, w_1) is a linearly independent eigenvector of the coupled system [17.9] for the simple eigenvalue $\lambda = (-\gamma)^{1/2}$. The other simple eigenvalue $-\lambda = -(-\gamma)^{1/2}$ exists by the symmetry of the coupled system [17.9] with the eigenvector $(v_1, -w_1)$.

The generalized eigenvalue problem [17.15] for unbounded self-adjoint differential operators A and K with strictly positive essential spectrum was studied by Chugunova and Pelinovsky [CHU 10] in Pontryagin's space [PON 44]. Here, we report the modification of the analysis needed to treat the case when the bottom of the essential spectrum of A touches zero. We shall first prove that a deformation of A to $A_\delta := A + \delta K$ for a small positive number δ shifts the essential spectrum away from zero.

LEMMA 17.2.– For small positive values of δ, there is a positive δ-independent constant d_0 such that

$$\sigma_e(A_\delta) \geq d_0 \delta. \qquad [17.20]$$

PROOF.– Since $\mathscr{M}$ and $\mathscr{L}$ are represented by the relatively compact perturbations of operators $\mathscr{M}_0$ and $\mathscr{L}_0$ with constant coefficients, we can use the Fourier transform $\mathscr{F}$ on $L^2(\mathbb{R})$ to compute:

$$\mathscr{F}(\mathscr{M}_0 + \delta \mathscr{L}_0^{-1}) = k^2 \hat{\mathscr{L}}_0(k) + \delta \hat{\mathscr{L}}_0^{-1}(k), \quad k \in \mathbb{R},$$

where $\hat{\mathscr{L}}_0(k) \geq c_0 > 0$ for some c_0 by H1.

Let k_δ denote the positive global minimum of this function. By coercivity of $k^2 \hat{\mathscr{L}}_0(k)$, the global minimum is achieved at a finite value of k for small positive values of δ and there is a δ-independent positive constant K_0 such that $k_\delta \in [0, K_0]$. But then, there is a δ-independent positive constant d_0 such that $\hat{\mathscr{L}}_0^{-1}(k_\delta) \geq d_0$ and $k^2 \hat{\mathscr{L}}_0(k) + \delta \hat{\mathscr{L}}_0^{-1}(k) \geq d_0 \delta$ for small positive δ.

By lemma 17.2, the essential spectrum of A_δ for a small positive δ is strictly positive. Also, the kernel of A_δ is empty for a small positive δ because if $f \in \mathrm{Ker}(A_\delta)$, then $Af = -\delta K f$ but the negative eigenvalues do not accumulate near zero, thanks to the decomposition $\mathscr{L} = \mathscr{L}_0 + K_{\mathscr{L}}$ with a relatively compact perturbation $K_{\mathscr{L}}$. Therefore, there exists a small positive number δ such that operator A_δ is continuously invertible in $\mathscr{H}$ and the generalized eigenvalue problem [17.15] is rewritten in the shifted form:

$$(A + \delta K) w = (\gamma + \delta) K w, \qquad w \in \mathscr{H}. \qquad [17.21]$$

By the spectral theory of self-adjoint operators, the Hilbert space $\mathscr{H}$ can be equivalently decomposed into two orthogonal sums of subspaces which are invariant with respect to the operators K and A_δ for small positive values of δ:

$$\mathscr{H} = \mathscr{H}_K^- \oplus \mathscr{H}_K^+ = \mathscr{H}_{A_\delta}^- \oplus \mathscr{H}_{A_\delta}^+, \qquad [17.22]$$

where notation $-(+)$ stands for invariant subspaces of these operators related to the negative (positive) spectrum.

Since P is a projection defined by the eigenspace of $\mathscr{L}$ and $K = P \mathscr{L}^{-1} P$, it is obvious that:

$$\dim(\mathscr{H}_K^-) = n(\mathscr{L}). \qquad [17.23]$$

On the other hand, the number of negative eigenvalues of $A = P \mathscr{M} P$ is related to the number of negative eigenvalues of $\mathscr{M}$. Compared to the standard count of negative eigenvalues in constrained Hilbert spaces (theorem 4.1 in [PEL 11]), the complication here is that the zero eigenvalue of $\mathscr{M}$ is embedded to the edge of the essential spectrum of $\mathscr{M}$. In addition, the zero eigenvalue of A is shifted under the perturbation δK in the operator $A_\delta = A + \delta K$. The following two lemmas give the count of negative eigenvalues of A denoted as $n(A)$ and the count of $\dim(\mathscr{H}^-_{A_\delta})$ for a small positive number δ.

LEMMA 17.3.– Under assumptions H1–H4 on $\mathscr{L}$, we have:

$$n(A) = n(P\mathscr{M}P) = n(\mathscr{M}) - n_0 = n(\mathscr{L}) - n_0, \qquad [17.24]$$

where $n_0 = 1$ if $\langle \mathscr{L}^{-1}\phi_0, \phi_0\rangle < 0$ and $n_0 = 0$ if $\langle \mathscr{L}^{-1}\phi_0, \phi_0\rangle > 0$[4].

PROOF.– We study the behavior of the function $F(\mu) = \langle (\mu - \mathscr{M})^{-1} f_0, f_0\rangle$, which is well-defined for all $\mu \in \mathbb{R}_- \backslash \sigma(\mathscr{M})$. By H4′, it has the limit as μ approaches zero from below:

$$\lim_{\mu\uparrow 0} F(\mu) = -\langle \mathscr{M}^{-1} f_0, f_0\rangle = -\langle \mathscr{L}^{-1}\phi_0, \phi_0\rangle \neq 0.$$

Hence, the assertion of the lemma holds by the standard proof of theorem 4.1 in [PEL 11] (where it is formulated and proved in a more general setting).

LEMMA 17.4.– Under assumptions H1–H4 on $\mathscr{L}$, for a small positive number δ, we have:

$$\dim(\mathscr{H}^-_{A_\delta}) = n(\mathscr{L}). \qquad [17.25]$$

PROOF.– Negative eigenvalues of $\sigma_d(A_\delta)$ are defined from the eigenvalue problem:

$$Af + \delta K f = \lambda f \quad f \in \mathscr{H}. \qquad [17.26]$$

We use the assumptions that $\mathrm{Ker}(A) = \mathrm{span}\{\phi_0\}$ and $\langle K\phi_0, \phi_0\rangle = \langle \mathscr{L}^{-1}\phi_0, \phi_0\rangle \neq 0$[5]. Since negative eigenvalues of A in lemma 17.3 are bounded away from zero, they persist for small positive values of δ. Let $\mathscr{H}_\delta$ denote the orthogonal complement of the subspace spanned by $n(A)$ eigenvectors corresponding to these negative eigenvalues of $A + \delta K$ for small positive values of δ.

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4 By the standard technique (theorem 4.1 in [PEL 11]), we also have $n(\tilde{P}\mathscr{L}\tilde{P}) = n(\mathscr{L}) - n_0$, where $\tilde{P}$ is an orthogonal projection from $L^2(\mathbb{R})$ to $[\mathrm{span}\{\phi_0\}]^\perp \subset L^2(\mathbb{R})$.

5 Note that $P\phi_0 = \phi_0$ because $\langle f_0, \phi_0\rangle = 0$ and P is an orthogonal projection to $[\mathrm{span}\{f_0\}]^\perp$.

At $\delta = 0$, we have $\phi_0 \in \mathscr{H}_{\delta=0}$. If $\langle K\phi_0, \phi_0 \rangle < 0$, then $A_\delta = A + \delta K$ is not positive definite on $\mathscr{H}_\delta$ for small positive δ. Therefore, there is at least one negative (isolated) eigenvalue of A_δ, which becomes the zero eigenvalue of A as $\delta \to 0$ (the zero eigenvalue of A is embedded at the edge of $\sigma_e(A)$). Moreover, this is the only small negative eigenvalue of A_δ for small positive δ[6]. Thus, we conclude that if $\langle K\phi_0, \phi_0 \rangle = \langle \mathscr{L}^{-1}\phi_0, \phi_0 \rangle < 0$, then:

$$\dim(\mathscr{H}^-_{A_\delta}) = n(A) + 1 = n(\mathscr{L}).$$

On the other hand, if $\langle K\phi_0, \phi_0 \rangle > 0$, the operator $A_\delta = A + \delta K$ is strictly positive on the subspace $\mathscr{H}_\delta$ for small positive δ. Therefore, in this case, we have:

$$\dim(\mathscr{H}^-_{A_\delta}) = n(A) = n(\mathscr{L}).$$

The assertion of the lemma is proven in both the cases.

We are now ready to use theorem 1 from [CHU 10]. Note that although the theorem was proven under the assumption that the essential spectrum of A is bounded away from zero, the shift of A to A_δ satisfying $\sigma_e(A_\delta) \geq d_0\delta > 0$ justifies the technique behind the proof of theorem 1 in [CHU 10] for a small positive number δ. To formulate the theorem, we introduce some notations for the numbers of particular eigenvalues γ of the generalized eigenvalue problem [17.15] with the account of their algebraic multiplicities.

– N_p^- (N_n^-) is the number of negative eigenvalues γ whose (generalized) eigenvectors are associated to the non-negative (non-positive) values of the quadratic form $\langle K\cdot, \cdot \rangle$.

– N_p^+ (N_n^+) is the number of positive eigenvalues γ whose (generalized) eigenvectors are associated to the non-negative (non-positive) values of the quadratic form $\langle K\cdot, \cdot \rangle$.

– N_p^0 (N_n^0) is the multiplicity of zero eigenvalue whose (generalized) eigenvectors are associated to the non-negative (non-positive) values of the quadratic form $\langle K\cdot, \cdot \rangle$.

– N_{c^+} (N_{c^-}) is the number of complex eigenvalues γ in the upper (lower) half-plane. Because A and K are real-valued, we have $N_{c^+} = N_{c^-}$.

6 The edge of $\sigma_e(A)$ may generate additional eigenvalues by means of edge bifurcations [KAP 02]. All these eigenvalues are strictly positive because they detach from the bottom of $\sigma_e(A_\delta)$ which is as small as $\mathscr{O}(\delta)$, whereas the distance of these eigenvalues from the bottom of $\sigma_e(A_\delta)$ may only change as a superlinear function of δ as $\delta \to 0$ [KAP 02]. Therefore, all these eigenvalues via edge bifurcations are necessarily positive.

We are now ready to reformulate theorem 1 from [CHU 10].

THEOREM 17.2.– Under assumptions H1–H4, for a small positive number δ, eigenvalues of the generalized eigenvalue problem [17.21] are counted as follows:

$$N_p^- + N_n^0 + N_n^+ + N_{c^+} = \dim(\mathscr{H}_{A_\delta}^-), \quad [17.27]$$

$$N_n^- + N_n^0 + N_n^+ + N_{c^+} = \dim(\mathscr{H}_K^-). \quad [17.28]$$

To apply theorem 17.2 to the count of isolated and embedded eigenvalues in the stability problem [17.2], we recall from [17.23] and [17.25] that $\dim(\mathscr{H}_K^-) = n(\mathscr{L})$ and $\dim(\mathscr{H}_{A_\delta}^-) = n(\mathscr{L})$. At the same time, definition of N_n^0 yields $N_n^0 = n_0$, where n_0 is introduced in lemma 17.3. Using these relations, we rewrite equalities [17.27] and [17.28] in the more explicit form:

$$N_p^- + N_n^+ + N_{c^+} = n(\mathscr{L}) - n_0, \quad [17.29]$$

$$N_n^- + N_n^+ + N_{c^+} = n(\mathscr{L}) - n_0. \quad [17.30]$$

We now need to compute numbers N_p^-, N_n^-, N_n^+ and N_{c^+} for real and complex eigenvalues of the generalized eigenvalue problem [17.15], which are related to real, imaginary and complex eigenvalues of the spectral problem [17.2]. Note that the imaginary eigenvalues of the spectral problem [17.2] may be embedded into the continuous spectrum of the operator $\partial_x \mathscr{L}$.

Again we recall here the Hamiltonian symmetry, that is if $\lambda \neq 0$ is a simple eigenvalue of the spectral problem [17.2], then $-\lambda$ is also a simple eigenvalue of the spectral problem [17.2] and both λ and $-\lambda$ correspond to the same double eigenvalue $\gamma = -\lambda^2 \neq 0$ of the generalized eigenvalue problem [17.15].

LEMMA 17.5.– Let $\lambda_j \in \mathbb{R}_+$ and $\tilde{\lambda}_j = -\lambda_j \in \mathbb{R}_-$ be simple eigenvalues of the spectral problem [17.2] associated with the real-valued eigenvectors v_j and $\tilde{v}_j$ in $D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R})$. Then, we have:

$$\langle \mathscr{L} v_j^\pm, v_j^\pm \rangle = \pm 2 \langle \mathscr{L} \tilde{v}_j, v_j \rangle, \quad \langle \mathscr{L} v_j^\pm, v_j^\mp \rangle = 0, \quad [17.31]$$

where $v_j^\pm = v_j \pm \tilde{v}_j$ are linearly independent.

PROOF.– We recall that the eigenvectors v_j and $\tilde{v}_j$ for distinct simple eigenvalues λ_j and $\tilde{\lambda}_j$ are linearly independent; hence, the linear combinations v_j^+ and v_j^- are linearly independent. Since $\lambda_j \neq 0$ and v_j is real-valued, we have:

$$\langle \mathscr{L} v_j, v_j \rangle = \frac{1}{\lambda_j} \langle \mathscr{L} v_j, \partial_x \mathscr{L} v_j \rangle = 0.$$

Similarly, $\langle \mathscr{L}\tilde{v}_j, \tilde{v}_j \rangle = 0$. The orthogonality relations [17.31] hold by direct computations.

LEMMA 17.6.– Let $\lambda_j \in i\mathbb{R}_+$ and $\bar{\lambda}_j = -\lambda_j \in i\mathbb{R}_-$ be simple eigenvalues of the spectral problem [17.2] associated with the eigenvectors v_j and $\bar{v}_j$ in $D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R})$. Then, we have:

$$\langle \mathscr{L} v_j^{\pm}, v_j^{\pm} \rangle = 2\langle \mathscr{L} v_j, v_j \rangle, \quad \langle \mathscr{L} v_j^{\pm}, v_j^{\mp} \rangle = 0, \qquad [17.32]$$

where $v_j^{\pm} = v_j \pm \bar{v}_j$ are linearly independent and $\langle \mathscr{L} v_j, v_j \rangle$ is real.

PROOF.– Since operator $\mathscr{L}$ is real-valued, the eigenvector v_j of the spectral problem [17.2] with $\mathrm{Im}(\lambda_j) \neq 0$ has both real and imaginary parts. Since $\lambda_j \neq 0$, we have:

$$\langle \mathscr{L}\bar{v}_j, v_j \rangle = \frac{1}{\lambda_j} \langle \mathscr{L}\bar{v}_j, \partial_x \mathscr{L} v_j \rangle = 0.$$

Furthermore, since $\mathscr{L}$ is self-adjoint, we have $\langle \mathscr{L} v_j, v_j \rangle = \langle \mathscr{L}\bar{v}_j, \bar{v}_j \rangle$. The orthogonality equations [17.32] hold by direct computations.

PROOF OF THEOREM 17.1.– By symmetries of the linearized Hamiltonian system, each eigenvalue $\gamma_j = -\lambda_j^2$ of the generalized eigenvalue problem [17.15] has a double multiplicity compared to the eigenvalue λ_j of the spectral problem [17.2]. From two linearly independent eigenvectors $v_j^{\pm} \in D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}(\mathbb{R}) \subset L^2(\mathbb{R})$ constructed in lemmas 17.5 and 17.6, we obtain two linearly independent eigenvectors $w_j^{\pm} = \partial_x^{-1} v_j^{\pm} \in \mathscr{H}$ of the generalized eigenvalue problem [17.15][7].

By the orthogonality condition [17.31], we have $N_n^- = N_p^-$ for a negative eigenvalue $\gamma_j = -\lambda_j^2$ corresponding to two real eigenvalues λ_j and $-\lambda_j$. Since $N_n^- + N_p^- = 2N_r$ because of the double multiplicity of eigenvalues γ_j compared to the multiplicity of eigenvalues λ_j, we obtain $N_n^- = N_p^- = N_r$. Similarly, for a complex eigenvalue $\gamma_j = -\lambda_j^2$ corresponding to two complex eigenvalues λ_j and $-\lambda_j$, we obtain $N_{c^+} = 2N_c$.

7 The same relations hold in the case of complex eigenvalues λ_j with $\mathrm{Re}(\lambda_j) \neq 0$ and $\mathrm{Im}(\lambda_j) \neq 0$. If v_j and $\tilde{v}_j$ linearly denote independent eigenvectors of the spectral problem [17.2] for complex eigenvalues λ_j and $-\lambda_j$, then we can define two linearly independent eigenvectors $w_j = \partial_x^{-1} v_j$ and $\tilde{w}_j = \partial_x^{-1} \tilde{v}_j$ of the generalized eigenvalue problem [17.15] in $\mathscr{H}$ for the double eigenvalue $\gamma_j = -\lambda_j^2$. In the case of complex eigenvalues, we do not care about the values of the quadratic form associated with the operator $\mathscr{L}$ computed at the eigenvectors.

By the orthogonality condition [17.32], the double multiplicity of the positive eigenvalue $\gamma_j = -\lambda_j^2$ corresponding to two imaginary eigenvalues λ_j and $\bar{\lambda}_j = -\lambda_j$, and the definition of N_i^-, we obtain $N_n^+ = 2N_i^-$. The equality [17.3] follows equivalently from either equality [17.29] or [17.30][8].

REMARK 17.2.– The count of eigenvalues provided by the equality [17.28] in theorem 17.2 is a sufficient tool to prove theorem 17.1 since it follows from definitions that $\dim(\mathscr{H}_K^-) = n(\mathscr{L})$, $N_n^0 = n_0$, whereas it follows from lemmas 17.5 and 17.6 that $N_n^- = N_r$, $N_n^+ = 2N_i^-$, and $N_{c^+} = 2N_c$. Along this avenue, the count of negative eigenvalues of operators $\mathscr{M}$, A and A_δ in lemmas 17.1, 17.3, and 17.4 (which is not so easy to prove) is redundant and unnecessary.

17.4. Generalization of theorem 17.1 for a periodic nonlinear wave

We shall now take remark 17.2 into account for an easy proof of the instability index count for periodic waves in the KdV-type evolution equations. These instability index counts were reported in [BRO 12a, DEC 10, HAR 08] by means of much longer and different analysis.

We now consider a $2L$-periodic nonlinear wave ϕ in a subspace $\mathscr{X}$ of Hilbert space $L^2_{\text{per}}(-L,L)$ equipped with the inner product $\langle\cdot,\cdot\rangle$[9] and an induced norm $\|\cdot\|$. The spectral stability of ϕ is still determined by the spectral problem [17.2], where $\mathscr{L} := E''(\phi)$ is a self-adjoint real-valued operator with a dense domain $D(\mathscr{L})$ in $L^2_{\text{per}}(-L,L)$. We assume that $\mathscr{L}$ has a compact resolvent so that the spectrum of $\mathscr{L}$ in $L^2_{\text{per}}(-L,L)$ is purely discrete. We reinforce assumptions H1, H2 and H3 in the slightly modified form:

H1). The spectrum of $\mathscr{L}$ is purely discrete and includes a finite number $n(\mathscr{L})$ of negative eigenvalues with eigenvectors in $D(\mathscr{L}) \subset L^2_{\text{per}}(-L,L)$.

H2). $\text{Ker}(\mathscr{L}) = \text{span}\{f_0\}$ with $f_0 \in D(\mathscr{L}) \cap D(\partial_x \mathscr{L} \partial_x) \cap \dot{H}^{-1}_{\text{per}}(-L,L)$ so that $\phi_0 = \partial_x^{-1} f_0 \in L^2_{\text{per}}(-L,L)$.

8 From the comparison between [17.27] and [17.28], which is justified by lemma 17.2 and the equality $N_p^- = N_n^-$, we obtain $\dim(\mathscr{H}_{A_\delta}^-) = \dim(\mathscr{H}_K^-) = n(\mathscr{L})$, which yields the second independent proof of lemma 17.4.

9 Note that we do not change notations for the inner product compared to the case $L^2(\mathbb{R})$ but understand that the integration is now performed on $[-L,L]$.

In addition, we note that $\mathrm{Ker}(\partial_x) = \mathrm{span}\{1\} \subset L^2_{\mathrm{per}}(-L,L)$ and define the matrix $\mathscr{D}$ as follows:

$$\mathscr{D} = \begin{bmatrix} \langle \mathscr{L}^{-1}\phi_0, \phi_0 \rangle & \langle \mathscr{L}^{-1}\phi_0, 1 \rangle \\ \langle \mathscr{L}^{-1}\phi_0, 1 \rangle & \langle \mathscr{L}^{-1}1, 1 \rangle \end{bmatrix}. \qquad [17.33]$$

Note that matrix $\mathscr{D}$ has finite elements because $\mathrm{span}\{1, \phi_0\} \perp \mathrm{Ker}(\mathscr{L})$ as it follows from the orthogonality conditions $\langle f_0, \phi_0 \rangle = \langle \partial_x \phi_0, \phi_0 \rangle = 0$ and $\langle f_0, 1 \rangle = \langle \partial_x \phi_0, 1 \rangle = 0$. We modify now assumption H4 as follows:

H3). Matrix $\mathscr{D}$ is invertible.

With the previous definitions of N_r, N_c, and N_i^-, the following theorem gives a modification of theorem 17.1 for a periodic nonlinear wave.

THEOREM 17.3.– Assume H1–H3. Then:

$$N_r + 2N_c + 2N_i^- = n(\mathscr{L}) - n(\mathscr{D}), \qquad [17.34]$$

where $n(\mathscr{D})$ is the number of negative eigenvalues of the matrix $\mathscr{D}$.

PROOF.– We extend the spectral problem [17.2] to the system of two coupled equations:

$$\mathscr{M} w = -\lambda v, \quad \mathscr{L} v = \lambda w, \qquad [17.35]$$

where $\mathscr{M} = -\partial_x \mathscr{L} \partial_x$, $v \in D(\partial_x \mathscr{L}) \cap \dot{H}^{-1}_{\mathrm{per}}(-L,L) \subset L^2_{\mathrm{per}}(-L,L)$, and $w \in D(\partial_x \mathscr{L} \partial_x) \subset L^2_{\mathrm{per}}(-L,L)$. The equivalence of simple eigenvalues of the coupled system [17.35] and those of the spectral problem [17.2] is proved similarly to in proposition 17.1.

By assumptions H1 and H2, the zero eigenvalue of $\mathscr{L}$ is isolated and simple. Let P be the orthogonal projection from $L^2_{\mathrm{per}}(-L,L)$ to $[\mathrm{span}\{f_0\}]^\perp \subset L^2_{\mathrm{per}}(-L,L)$. By a procedure that is similar to [17.16], [17.17] and [17.18], we obtain the generalized eigenvalue problem for a non-zero eigenvalue $\gamma \neq 0$:

$$Aw = \gamma K w, \quad w \in \mathscr{H}, \quad \mathscr{H} := D(\mathscr{M}) \cap [\mathrm{span}\{f_0\}]^\perp \subset L^2_{\mathrm{per}}(-L,L) \qquad [17.36]$$

where $A := P\mathscr{M}P$, $K := P\mathscr{L}^{-1}P$, and $\gamma := -\lambda^2$. The equivalence of double eigenvalues of the generalized eigenvalue problem [17.36] and pairs of simple eigenvalues of the coupled system [17.35] is proved similarly to in proposition 17.2.

The spectrum of $\mathscr{M}$ is purely discrete but the zero eigenvalue of $\mathscr{M}$ is now double since $\mathrm{Ker}(\mathscr{M}) = \mathrm{span}\{1, \phi_0\}$. Therefore, for a small positive number δ, assumptions of theorem 17.2 are satisfied and the equality [17.28] takes the form:

$$N_n^- + N_n^0 + N_n^+ + N_{c^+} = \dim(\mathscr{H}_K^-). \qquad [17.37]$$

By construction of $K = P\mathscr{L}^{-1}P$ and $\mathscr{H} = D(\mathscr{M}) \cap [\mathrm{span}\{f_0\}]^\perp \subset L^2_{\mathrm{per}}(-L,L)$, we have $\dim(\mathscr{H}_K^-) = n(\mathscr{L})$. On the other hand, N_n^0 denotes algebraic multiplicity of zero eigenvalues of the generalized eigenvalue problem [17.36] whose generalized eigenvectors are associated to non-positive values of the quadratic form $\langle K\cdot,\cdot\rangle$. Since $\mathrm{Ker}(\mathscr{M}) = \mathrm{span}\{1, \phi_0\}$ and the matrix $\mathscr{D}$ has no zero eigenvalue by assumption H3, we have $N_n^0 = n(\mathscr{D})$.

From analysis identical to lemmas 17.5 and 17.6, we also obtain $N_n^- = N_r$, $N_n^+ = 2N_i^-$ and $N_{c^+} = 2N_c$; hence, the equality [17.37] yields the instability index count [17.34] and the theorem is proven.

REMARK 17.3.– Equality [17.27] in theorem 17.2 can also be used for the correct instability index count, but this task would require the count of negative eigenvalues of operators $\mathscr{M}$, A and A_δ similar to that in lemmas 17.1, 17.3, and 17.4, which would result in the formula $\dim(\mathscr{H}^-_{A+\delta K}) = n(\mathscr{L})$ for a small positive number δ[10]. For the spectral problem associated with the KdV-type evolution equation, this equality is redundant because the spectral problem for the coupled system [17.35] is a squared version of the original spectral problem [17.2].

REMARK 17.4.– Eigenvectors of the spectral problem [17.2] in $L^2_{\mathrm{per}}(-L,L)$ for a non-zero eigenvalue λ are orthogonal to $\mathrm{Ker}(\mathscr{M}) = \mathrm{span}\{1, \phi_0\}$. Therefore, we can introduce a constrained space:

$$\tilde{\mathscr{H}} := D(\mathscr{L}) \cap [\mathrm{span}\{1, \phi_0\}]^\perp \subset L^2_{\mathrm{per}}(-L,L)$$

10 The identity $\dim(\mathscr{H}^-_{A+\delta K}) = n(\mathscr{L})$ follows from the identity $n(A) = n(P\mathscr{M}P) = n(\mathscr{L}) - n(\mathscr{D})$, which should hold despite the fact that the projection operator P is defined by the orthogonal complement of the one-dimensional subspace $\mathrm{Ker}(L) = \mathrm{span}\{f_0\}$. The corresponding argument goes as follows. To study $n(P\mathscr{M}P)$, we introduce a Lagrange multiplier ν and set up the self-adjoint spectral problem $\mathscr{M}w = \mu w + \nu f_0$ with the orthogonality condition $\langle f_0, w\rangle = 0$. Since $\mathscr{M} = -\partial_x \mathscr{L} \partial_x$, $f_0 = \partial_x \phi_0$ and $\mathrm{Ker}(\partial_x) = \mathrm{span}\{1\}$, we set $w = \partial_x g$, integrate in x and obtain the non-self-adjoint spectral problem $\mathscr{L}(-\partial_x^2)g = \mu g + \nu\phi_0 + \chi$ with two Lagrange multipliers ν and χ, under the constraints $\langle \phi_0, \partial_x^2 g\rangle = 0$ and $\langle 1, \partial_x^2 g\rangle = 0$. Smoothing it with a positive parameter ε and setting $g = (\varepsilon - \partial_x^2)^{-1}h$, we end up with the self-adjoint spectral problem $\mathscr{L}h = \mu(\varepsilon - \partial_x^2)^{-1}h + \nu\phi_0 + \chi$ under the constraints $\langle \phi_0, h\rangle = 0$ and $\langle 1, h\rangle = 0$, which can be studied using the standard technique (theorem 4.1 in [PEL 11]).

and a projection operator $\tilde{P} : L^2_{\text{per}}(-L,L) \to \tilde{\mathscr{H}}$ to reformulate the spectral problem [17.2] as a linearized Hamiltonian system with an invertible symplectic matrix:

$$\mathscr{L}_p v = \lambda J_p v, \quad v \in \tilde{\mathscr{H}}, \qquad [17.38]$$

where $\mathscr{L}_p := \tilde{P}\mathscr{L}\tilde{P}$ and $J_p := \tilde{P}\partial_x^{-1}\tilde{P}$. Standard analysis in constrained Hilbert spaces (theorem 4.1 in [PEL 11]) shows that $n(\mathscr{L}_p) = n(\mathscr{L}) - n(\mathscr{D})$. Now applying the general instability index count in the linearized Hamiltonian systems with an invertible symplectic operator [KAP 04], we can immediately obtain the instability index equality [17.34]. This proof of the instability index equality for the nonlinear periodic waves in Hamiltonian systems was introduced by Haragus and Kapitula [HAR 08].

Let us show how to recover the correct count of eigenvalues for the example of the focusing modified KdV equation:

$$u_t + 3u^2 u_x + u_{xxx} = 0. \qquad [17.39]$$

Traveling periodic waves in the form $u = \phi(x-ct)$ satisfies the differential equation:

$$\phi'' = c\phi - \phi^3, \qquad [17.40]$$

where the constant of integration is chosen to be zero. Two families of nonlinear periodic waves were considered by Deconinck and Kapitula [DEC 10] in the explicit form:

$$\phi(x) = \sqrt{2}k\,\text{dn}(x,k), \quad c = 2 - k^2, \qquad [17.41]$$

$$\phi(x) = \sqrt{2}k\,\text{cn}(x,k), \quad c = -1 + 2k^2, \qquad [17.42]$$

where dn and cn are Jacobi's elliptic functions and the period $L = 4K(k)$ is given by the complete elliptic integral of the first kind for a fixed $k \in (0,1)$.

Since $\mathscr{L} := -\partial_x^2 + c - 3\phi^2(x)$ with $\mathscr{L}\partial_x\phi = 0$ and $\mathscr{L}\partial_c\phi = -\phi$, assumption H2 is satisfied with $\phi_0 = \phi$. In addition, by scaling invariance of the stationary modified KdV equation [17.40], we can obtain:

$$\langle \mathscr{L}^{-1}\phi, \phi\rangle = -\frac{1}{2}\frac{d}{dc}\|\phi\|^2 < 0 \qquad [17.43]$$

and

$$\langle \mathscr{L}^{-1}\phi, 1\rangle = -\frac{d}{dc}\int_{-L}^{L}\phi(x)dx = 0. \qquad [17.44]$$

Let us denote:

$$F(k) := \langle \mathscr{L}^{-1}1, 1\rangle = \frac{1}{2L}\int_{-L}^{L}\mathscr{L}^{-1}(1)dx. \qquad [17.45]$$

For the dn-wave, explicit computations in [DEC 10] show that $n(\mathscr{L}) = 1$ and $F(k) > 0$ for all $k \in (0,1)$. Therefore, the instability index equality [17.34] with $n(\mathscr{L}) = n(\mathscr{D}) = 1$ yields spectral stability of the dn-wave for all $k \in (0,1)$.

For the cn-wave, explicit computations in [DEC 10] show that $n(\mathscr{L}) = 2$, whereas there is $k_* \approx 0.909$ such that $F(k) < 0$ for $0 < k < k^*$ and $F(k) > 0$ for $k^* < k < 1$. Therefore, the instability index equality [17.34] yields spectral stability of the cn-wave for $k \in (0,k_*)$ with $n(\mathscr{L}) = n(\mathscr{D}) = 2$ and spectral instability for $k \in (k_*,1)$ with $n(\mathscr{L}) = 2$, $n(\mathscr{D}) = 1$ and $N_r = 1$.

17.5. Conclusion

Although the instability index count is now well established both for solitary waves and periodic waves of the KdV-type evolution equations, there are still many directions of further development in the stability theory of nonlinear waves. In particular, Boussinesq equations involve the spectral problem [17.2] associated with a matrix differential operator $\mathscr{L}$ [LIN 08]. Although this matrix operator can be mapped to a self-adjoint form with a similarity transformation, it becomes difficult to transform both operators $\mathscr{L}$ and $\mathscr{M}$ in the coupled system [17.9] to the self-adjoint form. More direct approaches to the stability of solitary waves in Boussinesq systems can be found in recent works [HAK 13, STA 12]. The same complication may also occur in the system of coupled KdV-type equations.

Further extensions of the instability index count involves quadratic operator pencils, as well as general polynomial pencils, where the count of unstable eigenvalues becomes less precise. Work in this direction can be found in [BRO 12b, CHU 09, KOL 11].

17.6. Bibliography

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