Chapter 12

Continuum Hamiltonian Hopf Bifurcation I

In this chapter, Hamiltonian bifurcations in the context of noncanonical Hamiltonian matter models are described. First, a large class of 1 + 1 Hamiltonian multifluid models is considered. These models have linear dynamics with discrete spectra, when linearized about homogeneous equilibria, and these spectra have counterparts to the steady-state (SS) and Hamiltonian Hopf (HH) bifurcations when equilibrium parameters are varied. Examples of fluid sound waves and plasma and gravitational streaming are discussed in detail. Next, using these 1 + 1 examples as a guide, a large class of 2 + 1 Hamiltonian systems is introduced, and Hamiltonian bifurcations with continuous spectra are examined. It is shown how to attach a signature to such continuous spectra, which facilitates the description of the continuous Hamiltonian Hopf (CHH) bifurcation. This chapter lays the foundation for Kreĭn-like theorems associated with the CHH bifurcation that are more rigorously discussed in Chapter 13 (our companion chapter).

12.1. Introduction

A common bifurcation to instability, one that occurs in so-called natural Hamiltonian systems that have Hamiltonians composed of the sum of kinetic and potential energy terms, occurs when under a parameter change, the potential energy function changes from positive to negative curvature. In such a bifurcation, pairs of pure imaginary eigenvalues corresponding to real oscillation frequencies collide at zero and transition to pure real pairs, corresponding to growth and decay. This behavior, which can occur in general Hamiltonian systems and is termed the SS bifurcation, is depicted in the complex frequency $\omega = \omega_R + i\gamma$ plane in

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Figure 12.1(a). Alternatively, the HH bifurcation is the generic bifurcation that occurs in Hamiltonian systems when pairs of non-zero eigenvalues collide in the so-called Kreǐn collision [KRE 80] between eigenmodes of positive and negative signatures, as depicted in Figure 12.1(b). Such bifurcations occur in a variety of mechanical systems [CUS 90, VAN 85]; however, HH bifurcations also occur in infinite-dimensional systems with discrete spectra. In fact, one of the earliest such bifurcations was identified in the field of plasma physics [STU 58] for streaming instabilities, where signature was associated with the sign of the dielectric energy, and this idea made its way into fluid mechanics [CAI 79, MAC 86b]. Streaming instabilities were interpreted in the noncanonical Hamiltonian context in [MOR 90, KUE 95a], where signature was related to the sign of the oscillation energy in the stable Hamiltonian normal form [WIL 36, WEI 58] (see [12.15] below).

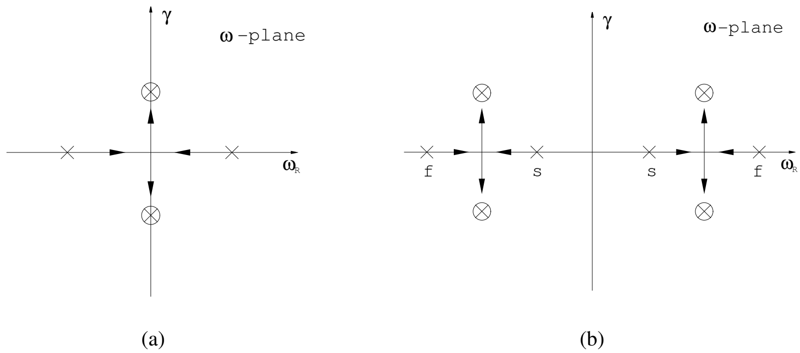


Figure 12.1. *Hamiltonian bifurcations with frequency $\omega = \omega_R + i\gamma$.*
a) Steady-state bifurcation with doublet bifurcating through the origin.
b) Hamiltonian Hopf bifurcation showing Kreǐn collision with quartet; slow modes (s) have opposite energy signature from fast (f)

The aim of this chapter and of Chapter 13 is to describe Hamiltonian bifurcations in the noncanonical Hamiltonian formalism (see [MOR 98]), which is the natural form for a large class of matter models including those that describe fluids and plasmas. Particular emphasis is on the CHH bifurcation, which is the terminology that we introduce for particular bifurcations that arise in Hamiltonian systems when there exists a continuous spectrum. There also exists a continuum steady-state (CSS) bifurcation, but this will only be mentioned briefly. A difficulty is faced when generalization of Kreǐn's theorem is attempted, which states that a necessary condition for the bifurcation to instability is that the colliding eigenvalues of the HH bifurcation have opposite signature to systems with continuous spectra. This difficulty arises because 'eigenfunctions' associated with the continuous spectrum are not normalizable, in the usual sense, and consequently, obstacles have to be overcome to define signature for the continuous spectrum. This was done first in the context of the Vlasov equation in [MOR 92, MOR 00] and for fluid shear flow in [BAL 01]. Given this definition of signature, it became possible in [HAG 11b, HAG 11a] to define the CHH, a meaningful generalization of the HH bifurcation.

In this chapter, we motivate and explore aspects of the CHH, which are discussed further in Chapter 13. To this end, we describe in sections 12.2 and 12.3 large classes of Hamiltonian systems that possess discrete and continuous spectra when linearized about equilibria. These classes are noncanonically Hamiltonian, as is the case in general for matter models in terms of Eulerian variables. For a general field variable Ψ that represents the state of such a system, a noncanonical Hamiltonian dynamical system has the form

$$\Psi_t = \{\Psi, H\} = \mathfrak{J}\,\frac{\delta H}{\delta \Psi}, \qquad [12.1]$$

where $H[\Psi]$ is the Hamiltonian functional and $\{\,,\,\}$ is the Poisson bracket defined by

$$\{F, G\} = \int d\mu\, \frac{\delta F}{\delta \Psi}\,\mathfrak{J}[\Psi]\,\frac{\delta G}{\delta \Psi}\,. \qquad [12.2]$$

In general, we may consider a $\mu + 1$ multicomponent theory, i.e. $\Psi(\mu, t) = (\Psi^1, \Psi^2, ...)$, with $\mathfrak{J}$ being an operator that makes [12.2] a Lie algebra realization on functionals (observables). Because the operator $\mathfrak{J}$ need not have the canonical form, may depend on Ψ and may possess degeneracy, this structure was referred to in [MOR 80b] as noncanonical. Because of the degeneracy, the Poisson bracket of [12.2] possesses Casimir invariants $C[\Psi]$ that satisfy

$$\{C, F\} \equiv 0 \qquad \forall\, F\,. \qquad [12.3]$$

We refer the reader to [MOR 98, MAR 99] for further details.

In section 12.2, we consider a class of 1+1 multifluid theories that possess discrete spectra when linearized about homogeneous equilibria. The linearization procedure along with techniques for canonization and diagonalization, i.e. transformation to conventional canonical form and transformation to the stable normal form, respectively, are developed. Then, specific examples are considered that display both SS and HH bifurcations. In section 12.3, we consider a class of 2+1 theories. The class is described and the CHH bifurcation for the particular case of the Vlasov–Poisson system is discussed. The relationship to the results of section 12.2 is shown by introducing the waterbag model, which is one way of discretizing the continuous spectrum, and motivates our definition of the CHH bifurcation. Finally, in section 12.4, we summarize and introduce the material that will be discussed in Chapter 13.

12.2. Discrete Hamiltonian bifurcations

We first describe a class of Hamiltonian theories of fluid type that have equilibria with discrete spectra. Three examples are considered that demonstrate the occurrence of Hamiltonian bifurcations like those of finite-dimensional systems. In the last example of section 12.2.2.3, the HH bifurcation is seen to arise in the context of streaming.

12.2.1. *A class of 1 + 1 Hamiltonian multifluid theories*

For our purposes, here, it is sufficient to consider a class of 1+1 theories of Hamiltonian fluid type. These theories have space-time independent variables (x, t), where $x \in \mathbb{T} \subset \mathbb{R}$, where $\mathbb{T} = [0, 2\pi)$, on which we assume spatial periodicity for dependent variables of fluid type, $\Psi = (\rho_1, \rho_2, \dots u_1, u_2, \dots)$, where $\rho_\alpha(x,t)$ and $u_\alpha(x,t)$ are the density and velocity fields, respectively, with $\alpha = 1, 2, \dots, M$. These fields will be governed by a coupled set of ideal fluid-like equations generated by a Hamiltonian with a noncanonical Poisson bracket.

The noncanonical Poisson bracket for the class is obtained from that for the ideal fluid [MOR 80b, MOR 82] reduced to one spatial dimension

$$\{F, G\} = \sum_{\alpha=1}^{M} \int_{\mathbb{T}} dx \left(\frac{\delta G}{\delta \rho_\alpha} \partial \frac{\delta F}{\delta u_\alpha} - \frac{\delta F}{\delta \rho_\alpha} \partial \frac{\delta G}{\delta u_\alpha} \right), \qquad [12.4]$$

where the shorthand $\partial := \partial/\partial x$ is used and $\delta F/\delta u_\alpha$ and $\delta F/\delta \rho_\alpha$ are the usual functional (variational) derivatives (see e.g. [MOR 98]). We consider Hamiltonian functionals of the following form:

$$H[\rho_\alpha, u_\alpha] = \sum_{\alpha=1}^{M} \int_{\mathbb{T}} dx \left(\frac{1}{2}\rho_\alpha u_\alpha^2 + \rho_\alpha U_\alpha(\rho_\alpha) + \frac{1}{2}\rho_\alpha \Phi \right), \qquad [12.5]$$

where the internal energy per unit mass, U_α, is arbitrary but often taken to be $U_\alpha = \kappa\rho^{(\gamma-1)}/(\gamma-1)$ where κ and the polytropic index γ are positive constants. The coupling between the fluids is included by means of a field Φ that satisfies

$$\Phi(x,t) = \sum_{\beta=0}^{M} \mathfrak{P}[\rho_\beta], \qquad [12.6]$$

where $\mathfrak{P}$ is a symmetric pseudo-differential operator, $\int_{\mathbb{T}} dx\, f\mathfrak{P}[g] = \int_{\mathbb{T}} dx\, g\mathfrak{P}[f]$, and an arbitrary constant term ρ_0 has been included on the right hand side of [12.6].

From [12.6], we obtain

$$\frac{\delta H}{\delta \rho_\alpha} = \frac{u_\alpha^2}{2} + h_\alpha + \Phi \quad \text{and} \quad \frac{\delta H}{\delta u_\alpha} = \rho_\alpha u_\alpha, \qquad [12.7]$$

where the enthalpy $h_\alpha = \partial(\rho_\alpha U_\alpha)/\partial\rho_\alpha$ and the pressure of each fluid is given by $p_\alpha = \rho_\alpha^2 \partial U_\alpha/\partial\rho_\alpha$. Using [12.7] with [12.4], gives

$$\begin{aligned} \frac{\partial \rho_\alpha}{\partial t} &= \{\rho_\alpha, H\} = -\partial(\rho_\alpha u_\alpha), \\ \frac{\partial u_\alpha}{\partial t} &= \{u_\alpha, H\} = -u_\alpha \partial u_\alpha - \partial p_\alpha/\rho_\alpha - \partial\Phi, \end{aligned}$$

which constitute a system of fluid equations coupled through Φ alone.

The noncanonical bracket of [12.4] is degenerate and possesses the following Casimir invariants

$$C_\alpha^\rho = \int_{\mathbb{T}} dx\, \rho_\alpha \quad \text{and} \quad C_\alpha^u = \int_{\mathbb{T}} dx\, u_\alpha, \quad \alpha = 1, \ldots, M\,. \qquad [12.8]$$

These invariants satisfy $\{C_\alpha^{u,\rho}, F\} \equiv 0$ for all functionals F. The physical significance of these Casimirs can be traced back to the Liouville theorem of kinetic theory [MOR 87] (see section 12.3.2.2).

12.2.1.1. *Equilibrium and stability*

Because of the existence of the Casimir invariants, the Hamiltonian is not unique and, consequently, equilibria possess a variational principle since

$$0 = \mathfrak{J}[\Psi]\,\frac{\delta H}{\delta \Psi} = \mathfrak{J}[\Psi]\,\frac{\delta F}{\delta \Psi}\,,$$

where $F = H + C$. That is, $\delta F/\delta\Psi = 0 \Rightarrow \Psi_t = 0$. In the present context, this amounts to $\delta F = \delta\left(H + \sum_\alpha \lambda^\rho_\alpha C^\rho_\alpha + \lambda^u_\alpha C^u_\alpha\right) = 0$, with Lagrange multipliers $\lambda^{\rho;u}_\alpha \in \mathbb{R}$, or

$$\frac{\delta F}{\delta\rho_\alpha} = \frac{u_\alpha^2}{2} + h_\alpha + \Phi + \lambda^\rho_\alpha = 0 \quad \text{and} \quad \frac{\delta F}{\delta u_\alpha} = \rho_\alpha u_\alpha + \lambda^u_\alpha = 0\,. \qquad [12.9]$$

Equations [12.9] have the equilibrium solution $\Phi_e \equiv 0$, and $\rho^e_\alpha \in \mathbb{R}^{>0}$ and $u^e_\alpha \in \mathbb{R}$.

Expansion around such equilibria gives a linear dynamical system. Because the equilibria of interest are homogeneous, we can use the following expression *en route* to linearization:

$$\rho_\alpha = \rho^e_\alpha + \sum_{k\in\mathbb{Z}} \rho^\alpha_k(t)\, e^{ikx} \quad \text{and} \quad u_\alpha = u^e_\alpha + \sum_{k\in\mathbb{Z}} u^\alpha_k(t)\, e^{ikx},$$

where the equilibrium constants $(\rho^e_\alpha, u^e_\alpha)$ could be absorbed into the sum by redefinition of the $k = 0$ terms. For linearization, we expand in the smallness of $(\rho^\alpha_k, u^\alpha_k)$.

Functionals of $(\rho^\alpha_k, u^\alpha_k)$ can be mapped onto functions of the Fourier components by insertion of the Fourier series, i.e.

$$F[\rho_\alpha, u_\alpha] = f(\rho^\alpha_0, \rho^\alpha_{\pm1}, \rho^\alpha_{\pm2}, \dots; u^\alpha_0, u^\alpha_{\pm1}, u^\alpha_{\pm2}, \dots),$$

and this transformation (for our purposes) can be considered invertible upon using

$$u^\alpha_k = \frac{1}{2\pi}\int_{\mathbb{T}} dx\, u_\alpha(x)\, e^{-ikx}\,.$$

Functional derivatives can also be expanded, e.g.

$$\frac{\delta F}{\delta u_\alpha} = \sum_{k\in\mathbb{Z}} \left(\frac{\delta F}{\delta u_\alpha}\right)_k e^{ikx} \quad \text{and} \quad \left(\frac{\delta F}{\delta u_\alpha}\right)_{-k} = \frac{1}{2\pi}\frac{\partial f}{\partial u^\alpha_k}, \qquad [12.10]$$

where the second equality follows from the chain rule (see, e.g., [TAS 11]). Using [12.10] and its counterpart for ρ_α, the bracket of [12.4] becomes

$$[f,g] = \sum_{k\in\mathbb{Z}} \sum_{\alpha=1}^{M} \frac{ik}{2\pi}\left(\frac{\partial g}{\partial u^\alpha_k}\frac{\partial f}{\partial \rho^\alpha_{-k}} - \frac{\partial f}{\partial u^\alpha_k}\frac{\partial g}{\partial \rho^\alpha_{-k}}\right). \qquad [12.11]$$

Observe that in the Poisson bracket of [12.11] the Casimir invariants of [12.8], the $k = 0$ components of (ρ_α, u_α), have been removed, i.e. the bracket has become non-degenerate in terms of the ostensible dynamical variables $(\rho_k^\alpha, u_k^\alpha)$. Geometrically, the choice of equilibrium selects the symplectic leaf on which the dynamics takes place.

It remains to determine the Hamiltonian. This is done by inserting the Fourier expansions of (ρ_α, u_α) into [12.5]. From this, we obtain the full nonlinear dynamics in terms of Fourier amplitudes, but since our interest is in bifurcations of the linear dynamics, we expand in the smallness of the amplitudes to obtain a quadratic form. Although this can be done in general terms, we prefer to explore particular cases of equilibrium and stability in section 12.2.2. However, before doing so, we make some general comments about canonization and diagonalization.

12.2.1.2. *Canonization and diagonalization*

The bracket of [12.11] is not yet in canonical form. To canonize, we rewrite the sums as follows:

$$
\begin{aligned}
[f, g] \;=\; & \sum_{k\in\mathbb{N}} \sum_{\alpha=1}^{M} \frac{ik}{2\pi} \left[\left(\frac{\partial g}{\partial u_k^\alpha} \frac{\partial f}{\partial \rho_{-k}^\alpha} - \frac{\partial f}{\partial u_k^\alpha} \frac{\partial g}{\partial \rho_{-k}^\alpha} \right) \right. \\
& \left. - \left(\frac{\partial g}{\partial u_{-k}^\alpha} \frac{\partial f}{\partial \rho_k^\alpha} - \frac{\partial f}{\partial u_{-k}^\alpha} \frac{\partial g}{\partial \rho_k^\alpha} \right) \right] \qquad [12.12] \\
=\; & \sum_{k\in\mathbb{N}} \sum_{m=1}^{2M} \left(\frac{\partial f}{\partial q_k^m} \frac{\partial g}{\partial p_k^m} - \frac{\partial g}{\partial q_k^m} \frac{\partial f}{\partial p_k^m} \right), \qquad [12.13]
\end{aligned}
$$

where in the second equality, the canonical fields (q_k^m, p_k^m) are obtained as particular real linear combinations of $(\rho_{\pm k}^\alpha, u_{\pm k}^\alpha)$. Thus, modes, i.e. degrees of freedom, are indexed by the wave number $k \in \mathbb{N}$ and an index m that takes two values for every value of the species index α. In terms of a choice of canonical fields, the Hamiltonian for the linear dynamics is given by the second variation as $\delta^2 F =: 2H_L$ and takes the form

$$
H_L = \sum_{k,\ell\in\mathbb{N}} \sum_{m,n=1}^{2M} z_k^m \, \mathfrak{A}_{mn}^{k\ell} \, z_\ell^n, \qquad [12.14]
$$

where $z_k^m := (q_k^m, p_k^m)$ and the matrix $\mathfrak{A}_{k\ell}^{mn}$ depends on the specific values of the equilibrium parameters $(\rho_\alpha^e, u_\alpha^e)$.

Given the system with Hamiltonian in the form of [12.14], it remains to effect a canonical transformation to a normal form. For example, if the system is linearly stable, then there exists a canonical transformation $(q_k^m, p_k^m) \leftrightarrow (Q_k^m, P_k^m)$, from real variables to real variables, where the Hamiltonian becomes the following in terms of the new canonical coordinates

$$H_L^s = \frac{1}{2} \sum_{k\in\mathbb{N}} \sum_{m=1}^{2M} \sigma_k^m\, \omega_k^m \Big((P_k^m)^2 + (Q_k^m)^2 \Big), \qquad [12.15]$$

where the frequencies $\omega_k^m \in \mathbb{R}^{>0}$ and the signature $\sigma_k^m \in \{\pm 1\}$. Thus, the stable normal form is just an infinite sum over simple harmonic oscillators. Those for which $\sigma_k^m = -1$ are negative energy modes, stable oscillations with negative energy (Hamiltonian). It is important to emphasize that even though the energy is negative, the system is stable. For finite-dimensional systems, the method for constructing the canonical transformation to normal coordinates (Q_k^m, P_k^m) is treated in standard texts and this method carries over. However, when negative energy modes exist, the method is somewhat complicated, and, although well-known in Hamiltonian dynamics lore, is not usually treated in physics texts. An accessible treatment is given in [TAS 11], where it is applied in a plasma physics context.

12.2.2. *Examples*

In order to make the ideas discussed in sections 12.2.1.1 and 12.2.1.2 more concrete, we consider a few examples that explicitly demonstrate canonization, diagonalization, and Hamiltonian bifurcations to instability in the context of multifluid models; in particular, the HH bifurcation will emerge for particular modes indexed by (k, α), just as it appears in finite-dimensional systems.

12.2.2.1. *Sound waves and multiplicity*

First, consider the case of a single fluid with an equilibrium state given by ρ_e, some positive constant, and $u_e \equiv 0$. The linear Hamiltonian is evidently

$$H_L = \frac{1}{2} \int_{\mathbb{T}} dx\, \left(\rho_e (\delta u)^2 + c_s^2 (\delta\rho)^2/\rho_e\right) = \pi \sum_{k\in\mathbb{Z}} \left(\rho_e |u_k|^2 + c_s^2 \left(|\rho_k|^2/\rho_e\right)\right),$$

where $c_s^2 = p_{\rho_e} = \rho_e(\rho U)_{\rho\rho}(\rho_e)$ is the sound speed. The appropriate Poisson bracket is that of [12.12] with a single α-term.

With some thought, canonization and diagonalization is possible in a single step, but we will proceed in a direct manner by assuming the canonical coordinates are

$$q_k^1 = \sqrt{2\pi}(u_k + u_{-k}) \quad \text{and} \quad q_k^2 = \sqrt{2\pi}(\rho_k + \rho_{-k}),$$

with corresponding momenta

$$p_k^1 = \frac{\sqrt{\pi/2}}{ik}(\rho_k - \rho_{-k}) \quad \text{and} \quad p_k^2 = \frac{\sqrt{\pi/2}}{ik}(u_k - u_{-k})\,.$$

A simple chain rule calculation takes [12.12] into the following

$$[f,g] = \sum_{k\in\mathbb{N}} \sum_{m=1}^{2} \left(\frac{\partial f}{\partial q_k^m}\frac{\partial g}{\partial p_k^m} - \frac{\partial g}{\partial q_k^m}\frac{\partial f}{\partial p_k^m} \right)\,. \qquad [12.16]$$

Observe in [12.16] that indexing a degree of freedom by $k \in \mathbb{N}$ requires a multiplicity index m. Each mode, which is described by an amplitude and a phase, constitutes a single degree of freedom; a single degree of freedom is thus two-dimensional, and consequently, each mode corresponds to two eigenvalues. For a stable degree of freedom, these eigenvalues correspond to two frequencies, one the negative of the other. Here, we have multiplicity, the reason for which will be mentioned when we diagonalize.

Now, using

$$u_k = \frac{1}{2\sqrt{2\pi}}\left(q_k^1 + 2ik\,p_k^2\right) \quad \text{and} \quad \rho_k = \frac{1}{2\sqrt{2\pi}}\left(q_k^2 + 2ik\,p_k^1\right),$$

valid with $k \to -k$, in the Hamiltonian H_L gives

$$H_L[q,p] = \frac{1}{4}\sum_{k\in\mathbb{N}}\Big(\rho_e|q_k^1|^2 + 4k^2c_s^2\,|p_k^1|^2/\rho_e + c_s^2|q_k^2|^2/\rho_e + 4k^2\rho_e\,|p_k^2|^2\Big).$$

The normal form is achieved upon substitution of the following canonical transformation:

$$q_k^1 = \sqrt{\frac{2kc_s}{\rho_e}}\,Q_k^1,\;\; p_k^1 = \sqrt{\frac{\rho_e}{2kc_s}}\,P_k^1,\;\; q_k^2 = \sqrt{\frac{2k\rho_e}{c_s}}\,Q_k^2,\;\; p_k^2 = \sqrt{\frac{c_s}{2k\rho_e}}\,P_k^2,$$

i.e., H_L becomes

$$H_L[Q,P] = \frac{1}{2}\sum_{k\in\mathbb{N}}\sum_{m=1}^{2} kc_s\Big((Q_k^m)^2 + (P_k^m)^2\Big)\,. \qquad [12.17]$$

This is the sought after normal form where the frequency of all modes is kc_s as appropriate for sound waves.

We close this example with a few comments. First, for a given wavelength as determined by $k \in \mathbb{N}$, there are in fact two modes: one that propagates to the right and one to the left. This is accounted for by the multiplicity index, m. In obtaining this normal form, we have assumed $c_s^2 = p_\rho > 0$, which can be traced back to a property of $U(\rho)$ and is in essence Le Châtelier's principle of thermodynamics, viz., that pressure increases upon compression. If we had some exotic fluid for which this was not the case, then the system would be unstable and the normal form of [12.17] would not be achievable. Imagine that the equilibrium parameter ρ_e can be varied and that at some critical value, c_s^2 makes a transition from positive to negative. Since a mode frequency $\omega = kc_s$, it is evident that this transition happens at zero frequency and, consequently, is a SS bifurcation (see Figure 12.1(a)). Moreover, because of the multiplicity, this is a degenerate bifurcation, where for each fixed k, four pure imaginary eigenvalues collide at zero frequency and then transition to four pure real eigenvalues of growing and decaying pairs. The situation is completely degenerate since this happens for all k values simultaneously. In section 12.2.3, we will see that the HH bifurcation, as depicted in Figure 12.1(b), can be transformed to a similar collision with four eigenvalues at zero frequency, but it differs in that after bifurcation, we obtain the Hamiltonian quartet, four eigenvalues with both real and imaginary parts, a situation that is sometimes called over stability.

12.2.2.2. *Counterstreaming ion beams with isothermal electrons*

Next, we consider a simple one-dimensional multifluid plasma configuration consisting of two cold counterstreaming ion beams in a neutralizing isothermal electron background. A detailed linear, nonlinear and numerical analysis of this problem, from a Hamiltonian perspective, can be found in [KUE 95a, KUE 95b], and we refer the reader to these references for further details.

The dynamical system of interest in dimensionless form is given by

$$
\begin{aligned}
&\frac{\partial u_\alpha}{\partial t} + u_\alpha \partial u_\alpha + \partial \phi = 0, \\
&\frac{\partial \rho_\alpha}{\partial t} + \partial \left(\rho_\alpha u_\alpha \right) = 0, \\
&\partial^2 \phi = e^\phi - \rho_+ - \rho_-,
\end{aligned}
\qquad [12.18]
$$

where $\alpha \in \{\pm\}$ labels each ion stream with velocity u_α. Here, ρ_α represents a dimensionless number density instead of mass density. Equation [12.18], Poisson's equation, is a constraint and in principle the electrostatic potential can be solved as $\phi\left(\rho_+, \rho_-\right)$ so that the entire system is described in terms of the dynamical variables

$\rho_\pm$ and $u_\pm$. As usual, the electric field is given by $E = -\partial\phi$. Thus, this system is of the class described in section 12.2.1, with [12.18] a specific case of [12.6]. It has the Hamiltonian functional

$$H = \int_{\mathbb{T}} dx \left(\sum_\alpha \frac{1}{2}\rho_\alpha u_\alpha^2 + \int_0^\phi d\phi\, \phi\, e^\phi + \frac{1}{2}\left(\partial\phi\right)^2 \right),$$

and Poisson bracket of [12.4].

Homogeneous equilibria follow from $\delta F = 0$ for $\lambda^\rho_\pm = -u_e^2/2$ and $\lambda^u_\pm = \mp u_e/2$, which are consistent with an equilibrium of ion streams of equal density and speed,

$$\rho_e^+ = \rho_e^- = \frac{1}{2}, \qquad u_e^+ = -u_e^- = u_e, \qquad E_e = \phi_e = 0,$$

that we assume for simplicity. Thus, we have a one-parameter family of equilibria controlled by u_e.

Linearizing about this equilibrium state gives the following Hamiltonian for the linear dynamics

$$\begin{aligned} H_L &= \frac{1}{2}\int_{\mathbb{T}} dx \left(\frac{1}{2}(\delta u_+)^2 + \frac{1}{2}(\delta u_-)^2 + 2\, u_e\, \delta\rho_+ \delta u_+ \right. \\ &\qquad \left. -2\, u_e\, \delta\rho_- \delta u_- + (\partial\delta\phi)^2 + (\delta\phi)^2 \right). \end{aligned}$$

Observe that the sign of H_L may be either positive or negative, depending on the perturbation; thus, we may have instability or negative energy modes in the system.

Expansion in a Fourier series as in section 12.2.1.1, including the expansion $\delta\phi = \sum_{k\in\mathbb{Z}} \phi_k e^{ikx}$, and using the linearized Poisson equation [12.18] gives $\phi_k = N_k/(1+k^2)$, where $N_k := \rho_k^+ + \rho_k^-$. With this expression, the energy H_L becomes

$$\begin{aligned} H_L &= \frac{\pi}{2}\sum_{k\in\mathbb{N}} \left(|u_k^+|^2 + |u_k^-|^2 \right. \\ &\qquad \left. +2u_e\big(\rho_k^+ u_{-k}^+ - \rho_k^- u_{-k}^- + \text{c.c.}\big) + 2\frac{|N_k|^2}{1+k^2} \right), \end{aligned} \qquad [12.19]$$

where c.c. denotes complex conjugate. Under the transformation

$$\begin{aligned}
\rho_k^+ &= \sqrt{\frac{u_e}{\pi}}\frac{k}{2}\,(p_2 - iq_1)\,, \qquad u_k^+ = \frac{1}{2\sqrt{\pi u_e}}\,(p_1 - iq_2) \\
\rho_k^- &= \sqrt{\frac{u_e}{\pi}}\frac{k}{2}\,(p_4 - iq_3)\,, \qquad u_k^- = \frac{1}{2\sqrt{\pi u_e}}\,(p_3 - iq_4)\,,
\end{aligned} \qquad [12.20]$$

with $\rho_{-k}^{\pm} = (\rho_k^{\pm})^*$ and $u_{-k}^{\pm} = (u_k^{\pm})^*$, the Poisson bracket becomes that of [12.13] with $M = 2$ and the linear Hamiltonian [12.19] becomes

$$H_L = \frac{1}{2}\sum_{k\in\mathbb{N}}\sum_{m,n=1}^{4}\left(p_k^m\,\mathfrak{M}_{mn}^k\,p_k^n + q_k^m\,\mathfrak{V}_{mn}^k\,q_k^n\right),$$

where

$$\mathfrak{M}^k = \begin{bmatrix} \frac{1}{2u_e} & ku_e & 0 & 0 \\ ku_e & \frac{k^2u_e}{1+k^2} & 0 & \frac{k^2u_e}{1+k^2} \\ 0 & 0 & \frac{1}{2u_e} & -ku_e \\ 0 & \frac{k^2u_e}{1+k^2} & -ku_e & \frac{k^2u_e}{1+k^2} \end{bmatrix}$$

$$\mathfrak{V}^k = \begin{bmatrix} \frac{k^2u_e}{1+k^2} & ku_e & \frac{k^2u_e}{1+k^2} & 0 \\ ku_e & \frac{1}{2u_e} & 0 & 0 \\ \frac{k^2u_e}{1+k^2} & 0 & \frac{k^2u_e}{1+k^2} & -ku_e \\ 0 & 0 & -ku_e & \frac{1}{2u_e} \end{bmatrix}.$$

Thus, in terms of the canonical coordinates of [12.20] the system is diagonal in k, but it remains to transform the 4×4 block structure, the part corresponding to the multiplicity, to normal form.

For values of u_e for which the system is stable, the diagonalizing canonical transformation is given explicitly in an appendix of [KUE 95a]. The reader is directed there to see how to obtain

$$H_L^s = \frac{1}{2}\sum_{k\in\mathbb{N}} \Big(\omega_k^+ \left((P_k^1)^2 + (Q_k^1)^2\right) - \omega_k^- \left((P_k^2)^2 + (Q_k^2)^2\right) + \omega_k^+ \left((P_k^3)^2 + (Q_k^3)^2\right) - \omega_k^- \left((P_k^4)^2 + (Q_k^4)^2\right)\Big). \quad [12.21]$$

Evidently, for each value of k, there exist four modes, two positive energy modes and two negative energy modes. The symmetry of the equilibrium facilitates the calculation of the frequencies, which are given by

$$\omega_k^{\pm} := k\left[\frac{1}{2(1+k^2)} + u_e^2 \pm \sqrt{\frac{1}{4(1+k^2)^2} + \frac{2u_e^2}{(1+k^2)}}\right]^{\frac{1}{2}} > 0, \quad [12.22]$$

which can be obtained from the plasma fluid dielectric (dispersion) function

$$\varepsilon(k,\omega) = 1 + \frac{1}{k^2} - \frac{1}{2}\left(\frac{1}{(\omega - ku_e)^2} + \frac{1}{(\omega + ku_e)^2}\right) = 0\,. \quad [12.23]$$

From [12.22] it is evident that all bifurcations to instability occur through zero frequency as depicted in Figure 12.1(a) and in fact are degenerate, i.e. if we fix k and vary u_e, then there is a value of u_e at which ω_-, the slow mode, vanishes and then becomes unstable with pure imaginary eigenvalues, two representing growth and two decay. Thus, this is another example of a SS bifurcation that is forced to be degenerate because of the imposed symmetry. In the next section, we will break this symmetry and obtain the HH bifurcation, but for variety, we do so in a physically different, yet mathematically similar, context.

12.2.2.3. *Jeans instability with streaming*

The widely studied Jeans instability occurs in Newtonian gravitational matter models. For the present example, we suppose matter is governed by our 1+1 fluid model with two interpenetrating streams. We refer the reader to [CAS 98] for background material and further details. The model is the same as that of section 12.2.2.2 except Poisson's equation is replaced by

$$\partial^2\phi = \rho_+ + \rho_- - \rho_\Lambda\,, \qquad [12.24]$$

where we incorporate Einstein's device of introducing a cosmological repulsion term, which in the Newtonian setting amounts to introducing a negative constant gravitational mass of density ρ_Λ. The sign change in [12.24] accounts for gravitational attraction.

The equilibrium for this case is similar to that of section 12.2.2.2, except we allow for asymmetry and, like the equilibrium of section 12.2.2.1, we allow for pressure in each stream. Specifically, we have the equilibrium constant densities ρ_e^+ and ρ_e^- such that $\rho_e^+ + \rho_e^- = \rho_\Lambda$ and $\phi_e = 0$, the two stream velocities $u_e^+ > 0$ and $-u_e^- > 0$, chosen in opposite directions, and two sound speeds $c_s^\pm$. Upon scaling, these can be reduced to four independent equilibrium parameters: u_e^+, u_e^-, $\beta := \rho_e^-/\rho_e^+$ and $c := c_s^-/c_s^+$.

From the results of sections 12.2.2.2 and 12.2.2.1, we can immediately write down the linearized Hamiltonian

$$\begin{aligned} H_L &= \frac{1}{2}\int_{\mathbb{T}} dx \Bigg(\rho_e^+(\delta u_+)^2 + \rho_e^-(\delta u_-)^2 + 2\,u_e^+\,\delta\rho_+\delta u_+ - 2\,u_e^-\,\delta\rho_-\delta u_- \\ &\qquad + (c_s^+)^2(\delta\rho_+)^2/\rho_e^+ + (c_s^-)^2(\delta\rho_-)^2/\rho_e^- - (\partial\delta\phi)^2 \Bigg)\,. \end{aligned}$$

Fourier expansion and canonization proceeds in the same manner as in the previous examples. In the case where the equilibrium parameters indicate stability, the diagonalization can be shown to give a Hamiltonian of the form of [12.15] with $M = 2$.

The frequencies are roots of the following "diagravic" function

$$\Gamma(k,\omega) = 1 + \frac{1}{2\left[(\omega - ku_e^+)^2 - k^2\right]} + \frac{\beta}{2\left[(\omega + cku_e^-)^2 - c^2k^2\right]} = 0\,, \qquad [12.25]$$

with two fast modes being positive energy modes and two slow modes being negative energy modes (see Figures 1 and 3 of [CAS 98]). In general, all four modes are distinct, but if we symmetrize parameters as in section 12.2.2.2, then the quadratic obtained from [12.25] becomes biquadratic and is easily solved, indicating degenerate modes of each sign as before. Evidently, this system possesses a rich parameter space, and various bifurcations to instability for various k-values are possible. In addition to the four parameters above, we can use k as a control parameter: we have scaled the system size to 2π, but upon reinstatement, this translates into varying k. The Jeans instability is a long wavelength instability, and we can observe the transition to instability as k decreases. This is immediate if $c = \beta = 1$ and $u_e = 0$, in which case [12.25] implies $\omega^2 = k^2 - 1$. Using k as the control parameter, as the wavelength is increased, we see the instability set in as a degenerate SS bifurcation. The situation is complicated with the presence of two streams, the subject of this section, and the HH bifurcation as depicted in Figure 12.1(b) is clearly present (see Figure 2 of [CAS 98]). This is quite generally the case for fluid systems with steaming equilibria. In section 12.3.2.2, we will see how multifluid streaming relates to the waterbag distribution of kinetic theory, and we will discuss explicitly the HH bifurcation in this context.

12.2.3. *Comparison and commentary*

It is evident from the discussion of section 12.2.2 that a requisite for determining an HH bifurcation is the identification of the energy for the linear system. In the context of noncanonical Hamiltonians systems, this naturally comes from second variation $\delta^2 F$, the Hamiltonian for the linear system. Sometimes "energy" expressions are obtained by direct manipulation of the linear equations of motion as done, for example, in the original MHD energy principle paper [BER 58], but this procedure can obscure the notion of signature. For example, a system of two simple harmonic oscillators conserves $\omega_1(q_1^2 + p_1^2) \pm \omega_2(q_2^2 + p_2^2)$ for both signs and either might be obtained by manipulation of the equations of motion. The unambiguous sign for the correct energy is uniquely given by $\delta^2 F$; this is important because this sign can drastically affect the behavior of the system when dissipation or nonlinearity is considered. For example, a system with a negative energy mode can become unstable to arbitrarily small deviations from the equilibrium when nonlinearity is added (see Cherry's example as described in [MOR 90, MOR 89]).

In the plasma literature, other definitions of energy are usually considered, e.g. in the context of streaming instabilities, the dielectric energy, which is proportional to $\omega|E|^2\partial\varepsilon/\partial\omega$, where E is the electric field amplitude, is incorporated. This expression was originally derived by von Laue [VON 05] for the energy content in a dielectric medium by tracking the energy input due to an external agent. However, we have seen how it arises from $\delta^2 F$, and only then can we be assured that it represents a quantity conserved by the linear dynamics. In fact, for our general multifluid model, the $\varepsilon(k,\omega)$ takes the form $\varepsilon(k,\omega) = 1 + \sum_\alpha \chi_\alpha(k,\omega)$, with a contribution χ_α from each fluid, e.g. that for counterstreaming and Jeans are [12.23] and [12.25], respectively, (also see [12.37] below) and it can be seen in general that $\delta^2 F \sim \omega|E|^2\partial\varepsilon/\partial\omega$. For neutral modes embedded in the continuous spectrum of Vlasov theory (see Chapter 13), the formula $\omega|E|^2\partial\varepsilon/\partial\omega$ remains valid [MOR 94, SHA 94], but this formula is incorrect for excitation of the continuous spectrum as shown in [MOR 92], where the correct alternative formula was first derived, and the notion of signature for the continuous spectrum was defined.

Sometimes, energy is defined in terms of the Lagrangian displacement variable as was done by manipulation of the linear equations of motion in [BER 58, FRI 60]. Such expressions can also be obtained by expansion of an appropriate Hamiltonian, $\delta^2 H$. It was shown in [MOR 90, MOR 98] that this procedure gives an expression that is essentially equivalent to $\delta^2 F$. See [AND 13] for a recent general discussion in the context of MHD.

In conventional Kreǐn theory, the signature is defined in terms of the Lagrange bracket (see, e.g., [MOS 58]). However, it is a simple matter to see that this corresponds to the normal form definition [MAC 86a, MOR 90], which follows by comparison of terms in the diagonalization procedure (see [WHI 37, TAS 11]). In [MOR 90], it was argued that all these definitions of signature, using the dielectric energy, $\delta^2 F$, $\delta^2 H$, and the Lagrange bracket, are essentially the same when they are meaningful.

One ostensible difference between the HH and SS bifurcations is that the latter occurs at zero frequency. However, one can affect a time-dependent canonical transformation so that all four HH eigenvalues of Figure 12.1(b) collide at zero frequency. To see this, consider one of the stable degrees of freedom, which has a contribution to the Hamiltonian in action-angle variables (θ_f, J_f) given by $H_f = \omega_f J_f$, where ω_f depends on the bifurcation control parameter and takes the value ω^* at the bifurcation point. Using the mixed variable generating function F_2 to transform to new canonical variables $(\Theta, \mathcal{J})$:

$$F_2(\theta_f, \mathcal{J}_f) = (\theta_f - \omega^* t)\mathcal{J}, \qquad [12.26]$$

we obtain

$$\Theta = \frac{\partial F_2}{\partial \mathcal{J}} = \theta_f - \omega^* t \quad \text{and} \quad J = \frac{\partial F_2}{\partial \theta_f} = \mathcal{J},$$

which amounts to moving into a rotating coordinate system with new Hamiltonian

$$\bar{H} = H + \frac{\partial F_2}{\partial t} = (\omega_f - \omega^*)\mathcal{J}\,.$$

Thus, in the new frame, the frequency $\omega_f - \omega^*$ vanishes at the bifurcation point. At bifurcation, the companion mode ω_s has the same value ω^*; consequently, a similar transformation will bring this mode to zero frequency at bifurcation. At first glance, we might think this has made the HH bifurcation identical to a degenerate SS bifurcation, but the behavior of the two beyond the bifurcation point is different. The degenerate SS bifurcation transitions to two purely growing and two purely decaying eigenvalues, while the HH transforms to over stability, i.e. it obtains a quartet structure immediately upon bifurcation. (We could argue that the frame shift could be a function of the control parameter, but with this line of reasoning all bifurcations could be made to look like SS bifurcations, even in the nonlinear regime.) The frame shift of [12.26] is identical to a Galilean shift that can be done for fluid and plasma theories in order to bring modes to zero frequency at bifurcation. This artifice is used in the development of the single-wave model [BAL 13] and will be considered in Chapter 13.

The connection between degeneracy and symmetry is well known, and there is a very large literature on bifurcations in finite-dimensional Hamiltonian systems with symmetry (see, e.g. [DEL 92] for an entryway). In our examples above we have seen, as expected, that this is also the case for infinite systems with discrete spectra. In fact, it is quite common for the dispersion relation to factor as a consequence of symmetry [TAS 08]. However, systems with symmetry and continuous spectra are less well-studied, but counterparts exist, e.g., the degeneracy of the SS bifurcation of Jeans inability with $u_e = 0$ of section 12.2.2.3 has a CSS counterpart when described by the Vlasov system (see section IIID of [BAL 13]).

12.3. Continuum Hamiltonian bifurcations

Now, we turn to the general class of 2+1 Hamiltonian mean-field theories in which the linear theories around equilibria possess a continuous spectrum. This is followed by the exposition of the two-stream instability in the Vlasov–Poisson equation, which is a standard example of the CHH bifurcation. Next, we introduce the waterbag reduction of the Vlasov–Poisson equation and use it to connect the two-stream instability to Kreǐn–bifurcations in the corresponding waterbag model, linking this section to section 12.2.

12.3.1. *A class of 2 + 1 Hamiltonian mean field theories*

We begin with the class of 2+1 Hamiltonian field theories introduced in [MOR 03], which have with a single dynamical variable, $f(q, p, t)$, a time-dependent density on the phase space variables $z := (q, p)$. The density satisfies a transport equation

$$\frac{\partial f}{\partial t} + [f, \mathcal{E}] = 0, \qquad [12.27]$$

where the bracket $[f, g] = f_q g_p - g_q f_p$ is the Poisson bracket for a single particle, and the particle energy $\mathcal{E}$ depends globally on f. Equation [12.27] is therefore a mean field theory, where f is a density of particles in phase space that generates $\mathcal{E}$ and is advected along the single particle trajectories that result from $\mathcal{E}$. The resulting equations are typically quasi-linear partial integro-differential equations. We assume that the particle energy arises from a Hamiltonian functional of the form $H[f] = H_1 + H_2 + H_3 + \ldots$, where generally H_n is the n-point energy, e.g.

$$H_1[\zeta] = \int_{\mathcal{Z}} d^2z\, h_1(z)\, f(z), \quad H_2[\zeta] = \frac{1}{2}\int_{\mathcal{Z}} d^2z \int_{\mathcal{Z}} d^2z' f(z)\, h_2(z, z')\, f(z'),$$

with h_1 and h_2 being interaction kernels. Here, we will only consider Hamiltonian systems with up to binary interactions, and we will assume that that h_2 possesses the symmetry $h_2(z, z') = h_2(z', z)$. If $\mathcal{E}$ is obtained from the field energy by functional differentiation

$$\mathcal{E} := \frac{\delta H}{\delta f} = h_1 + \int_{\mathcal{Z}} d^2z'\, h_2(z, z')\, f(z'),$$

then $H[f]$ is a constant of motion for [12.27].

Equation [12.27] with $\mathcal{E} = \delta H/\delta f$ is a Hamiltonian field theory [MOR 03] in terms of the noncanonical Lie–Poisson bracket of [MOR 80a, MOR 82]

$$\{F, G\} = \int_{\mathcal{Z}} d^2z\, f\left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]. \qquad [12.28]$$

This bracket depends explicitly upon f, unlike usual Poisson brackets that only depend on (functional) derivatives of the canonical variables. The bracket of [12.28] is antisymmetric and satisfies the Jacobi identity, though it is degenerate, unlike canonical brackets. The equations of motion may be written as

$$\frac{\partial f}{\partial t} = \{f, H\} = -\left[f, \frac{\delta H}{\delta f}\right] = -[f, \mathcal{E}], \qquad [12.29]$$

where $H = H_1 + H_2 + \ldots$.

As mentioned in section 12.1, degeneracy of the Poisson bracket gives rise to Casimir invariants, quantities that are conserved for *any* Hamiltonian. For the bracket of [12.28] the Casimir invariants are $C[f] = \int_{\mathcal{Z}} d^2z\, \mathcal{C}(f)$, where $\mathcal{C}(\zeta)$ is an arbitrary function. The existence of Casimir invariants leads to a foliation of phase space (in this case, a function space) with symplectic leaves, which are the level sets of the Casimir invariants and which inherit a symplectic structure from the Lie–Poisson bracket. The evolution of f is restricted to one of these symplectic leaves, and the equations on a single leaf are canonical.

In addition to the Casimir invariants and the total energy, there may be conserved momenta $P[f]$ generally arising from translation symmetries of the interaction kernels $h_1, h_2, \dots$. The system conserves momentum if there exists a canonical transformation of the phase space $\mathcal{Z}$, $z = (q,p) \longleftrightarrow \bar{z} := (\theta, I)$ such that in the new particle coordinates $\bar{z} := (\theta, I)$, the interactions h_1, h_2, etc., have upon composition with $z(\bar{z})$, one of the following two forms:

$$h_1 \circ z = \bar{h}_1(I), \qquad h_2 \circ (z, z') = \bar{h}_2(I, I', |\theta - \theta'|) \tag{12.30}$$

or

$$h_1 \circ z = 0, \qquad h_2 \circ (z, z') = \bar{h}_2(|I - I'|, |\theta - \theta'|)\,. \tag{12.31}$$

In the first case,

$$P[f] = \int_{\mathcal{Z}} d^2z\, I\, f(z)$$

is conserved, while in the second case, we have two kinds of translation invariance and thus two components of the momentum

$$P_1[f] = \int_{\mathcal{Z}} d^2z\, I\, f(z) \quad \text{and} \quad P_2[f] = \int_{\mathcal{Z}} d^2z\; \theta\, f(z)\,.$$

These momenta can be very useful (see e.g., [BAL 01]), but they will not be discussed further here.

For equilibrium states f is a function of the single particle constants of motion only, i.e. the single particle energy $\mathcal{E}$ and possibly momenta. The example treated here has an equilibrium that only depends on I, where (θ, I) are the action-angle variables corresponding to a given $\mathcal{E}$. For this reason, we set $f(\theta, I, t) = f_0(I) + \zeta(\theta, I, t)$ and then when a choice of f_0 is made, $\zeta(\theta, I, t)$ represents the main dynamical variable. The phase space is $\mathcal{Z} = \mathcal{D} \times \mathbb{T}$, i.e. periodic in $\theta \in [0, 2\pi) = \mathbb{T}$ and $I \in \mathcal{D}$ where $\mathcal{D} = \mathbb{R}$. Upon substitution of $f = f_0 + \zeta$ into $\mathcal{E}$, both of the forms of [12.30] and [12.31] can be written as follows:

$$\mathcal{E}[f_0 + \zeta] = \mathcal{E}[f_0] + \mathcal{E}[\zeta] =: h(I) + \Phi(\theta, I),$$

with

$$\Phi(\theta, I) = \mathcal{K}\zeta := \int_{\mathcal{D}} \mathrm{d}I' \int_{\mathbb{T}} \mathrm{d}\theta' \, K(I, I', |\theta - \theta'|) \, \zeta(\theta', I', t),$$

where h and K are determined by h_1 and h_2. Thus the governing equation is

$$\zeta_t + [f_0, \Phi] + [\zeta, h + \Phi] = 0, \qquad [12.32]$$

where $[f, g] = f_\theta g_I - g_\theta f_I$ and $\Omega(I) = h'$. Equation [12.32] will serve as a starting point for our subsequent linear analyses.

12.3.2. *Example of the CHH bifurcation*

All of the models described in section 12.3.1 possess CHH bifurcations; however, here, we will concentrate on the Vlasov–Poisson system. First, we describe it, then we make connections to the multifluid results of section 12.2.1 and in this way relate the CHH to the ordinary HH bifurcation.

12.3.2.1. *Vlasov–Poisson system*

The Vlasov–Poisson equation arises out of [12.27] through definition of the single particle energy $\mathcal{E}$ and potential ϕ, where $\mathcal{E} = p^2/2 + \phi$ and

$$f_t = -[f, \mathcal{E}] = -p f_q + \phi_q f_p \quad \text{and} \quad \phi_{qq} = 1 - \int_{\mathbb{R}} f dv. \qquad [12.33]$$

The interaction kernels for this model are: $h_1 = p^2/2$ and $h_2 = |q - q'|$.

The function f represents the density of a positive charge species in phase space, under the assumption that there is a neutralizing background with uniform negative charge density. The particles interact with each-other through electrostatic forces, which are included by the Poisson equation. Under the identification $q = \theta$, $p = I$, we recover [12.30]. Arbitrary functions of p alone, $f(q, p) = f_0(p) \equiv f_0(I)$, form an important class of solutions to this model, the spatially homogeneous equilibria. The analog of [12.32] is

$$\zeta_t + p\zeta_q - f_0' \phi_q - \zeta_p \phi_q = 0 \quad \text{and} \quad \phi_{qq} = -\int_{\mathbb{R}} dp \, \zeta \, .$$

Upon linearizing the Vlasov–Poisson system around a homogeneous stable equilibrium, i.e. dropping the nonlinear term $\zeta_p \phi_q$, and then supposing $\zeta = \zeta_k e^{ikq}$ to eliminate q (which is θ in the previous language) in lieu of the wave number k, $\zeta_k(p,t)$ becomes our dynamical variable that satisfies

$$\zeta_{k,t} = -ikp\zeta_k + ikf_0'\phi_k, \quad \text{and} \quad \phi_k = k^{-2}\int_{\mathbb{R}} \mathrm{d}\bar{p}\, \zeta_k(\bar{p},t),$$

which simplifies to the following integro-differential equation for ζ_k:

$$\zeta_{k,t} = -ikp\zeta_k + if_0'\, k^{-1}\int_{\mathbb{R}} d\bar{p}\, \zeta_k(\bar{p},t) =: -\mathfrak{T}_k\zeta_k\,. \qquad [12.34]$$

Here, we have introduced the time evolution operator $\mathfrak{T}_k$, whose spectrum under changes in f_0 we will study to understand the CHH bifurcation.

The linearized equations inherit a Hamiltonian structure. Because of the noncanonical form, linearization requires expansion of the Poisson bracket as well as the Hamiltonian. In terms of the variables ζ_k and ζ_{-k}, the Hamiltonian is

$$H_L[\zeta_k, \zeta_{-k}] = \frac{1}{2}\sum_{k\in\mathbb{N}} \left(-\int_{\mathbb{R}} dp\, \frac{p}{f_0'}\, |\zeta_k|^2 + |\phi_k|^2 \right),$$

with the Poisson bracket

$$\{F,G\}_L = \sum_{k\in\mathbb{N}} ik \int_{\mathbb{R}} dv\, f_0' \left(\frac{\delta F}{\delta\zeta_k}\frac{\delta G}{\delta\zeta_{-k}} - \frac{\delta F}{\delta\zeta_{-k}}\frac{\delta G}{\delta\zeta_k} \right). \qquad [12.35]$$

Observe from [12.35] that $k \in \mathbb{N}$, and thus, ζ_k and ζ_{-k} are independent variables that are almost canonically conjugate. Thus the complete system is

$$\zeta_{k,t} = -\mathfrak{T}_k\zeta_k \qquad \text{and} \qquad \zeta_{-k,t} = -\mathfrak{T}_{-k}\zeta_{-k},$$

whence it can be shown directly that the spectrum is Hamiltonian.

Now, we consider properties of the evolution operator $\mathfrak{T}_k$ defined by [12.34]. We suppose ζ_k varies as $\exp(-i\omega t)$, where ω is the frequency and $i\omega$ is the eigenvalue. For convenience, we also use $u := \omega/k$, where we can view $k \in \mathbb{R}^{>0}$ by varying the system size. The system is said to be spectrally stable if the spectrum of $\mathfrak{T}_k$ is less than or equal to zero or the frequency is always in the closed lower half plane. Since the system is Hamiltonian, the question of stability reduces to deciding if the spectrum is confined to the imaginary axis. The solutions of a spectrally stable system are guaranteed to grow at most subexponentially.

The operator $\mathfrak{T}_k$ is the sum of a multiplication operator and an integral operator, and the multiplication operator causes the continuous spectrum to be composed of the entire imaginary axis except possibly for some discrete points. Instability comes from the point spectrum. The linearized Vlasov–Poisson equation is not spectrally stable when the time evolution operator has an element of the point spectrum away from the imaginary axis (implying a doublet or quartet of modes with non-trivial real part). The point spectrum is composed of the roots of the plasma dispersion function

$$\varepsilon(k,u) := 1 - \frac{1}{k^2}\int_{\mathbb{R}} dp \frac{f_0'}{p-u}\,.$$

Here, $u = \omega/k$. The one-dimensional linearized Vlasov–Poisson system with homogeneous equilibrium f_0 is spectrally unstable if for some $k \in \mathbb{R}^{>0}$ and u in the upper half plane, the plasma dispersion function vanishes.

Using the Nyquist method that relies on the argument principle of complex analysis, Penrose [PEN 60] was able to relate the vanishing of $\varepsilon(k,u)$ to the winding number of the closed curve determined by the real and imaginary parts of ε as u runs along the real axis. Such closed curves are called Penrose plots. The crucial quantity is the integral part of ε as u approaches the real axis from above

$$\lim_{u\to 0^+} \frac{1}{\pi}\int_{\mathbb{R}} dp\, \frac{f_0'}{p-u} = \mathcal{H}[f_0'](u) + i f_0'(u),$$

where $\mathcal{H}[f_0']$ denotes the Hilbert transform, $\mathcal{H}[f_0'] = \frac{1}{\pi}⨍ dp\, f_0'/(p-u)$, where $⨍ := PV\int_{\mathbb{R}}$ indicates the Cauchy principle value, leading to the following expression for the contour, parametrized by $u \in \mathbb{R}$, in the complex plane:

$$\varepsilon(k,u) := 1 - \pi k^{-2}\mathcal{H}[f_0'](u) - i\pi k^{-2} f_0'(u)\,.$$

The image of the real line under this mapping is the Penrose plot, and its winding number about the origin is the number of members of the point spectrum of $\mathfrak{T}_k$ in the upper half plane.

Figure 12.2 shows the derivative of the distribution function, f_0', for the case of a Maxwellian distribution $f_0 = e^{-p^2}$ and Figure 12.3 shows the contour $-\mathcal{H}[f_0'] - i f_0'(u)$ that emerges from the origin in the complex plane at $u = -\infty$, descends and then wraps around to return to the origin at $u = \infty$. From this figure, it is evident that the winding number of the $\varepsilon(k,u)$-plot is zero for any fixed $k \in \mathbb{R}$, and as a result, there are no unstable modes. Here, we take the value of k to be fixed.

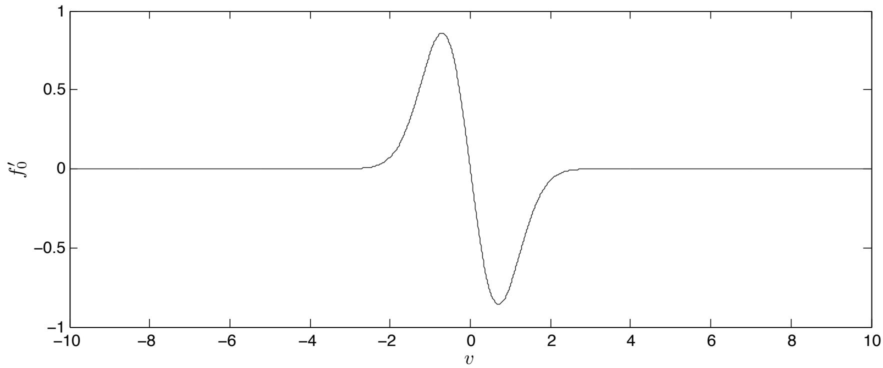


Figure 12.2. f_0' *for a Maxwellian distribution function*

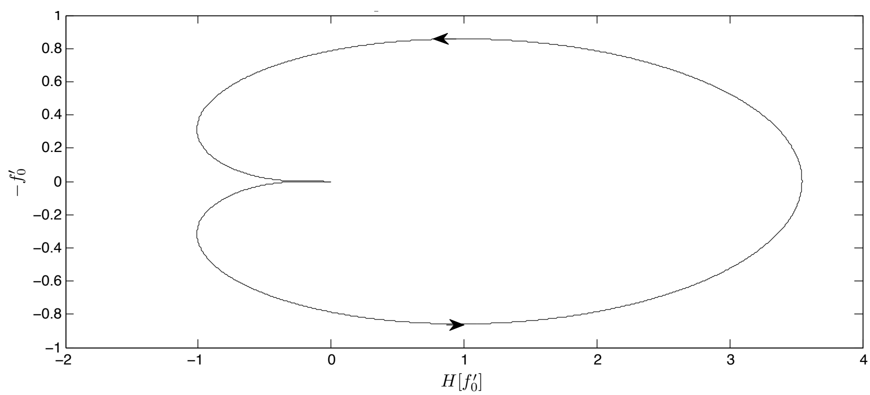
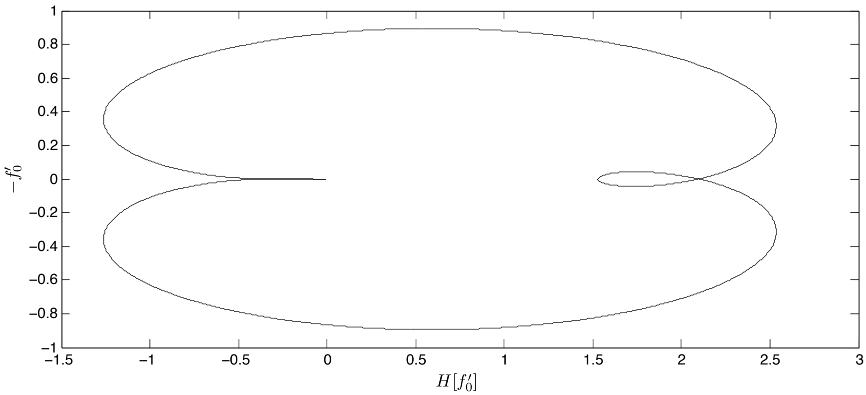


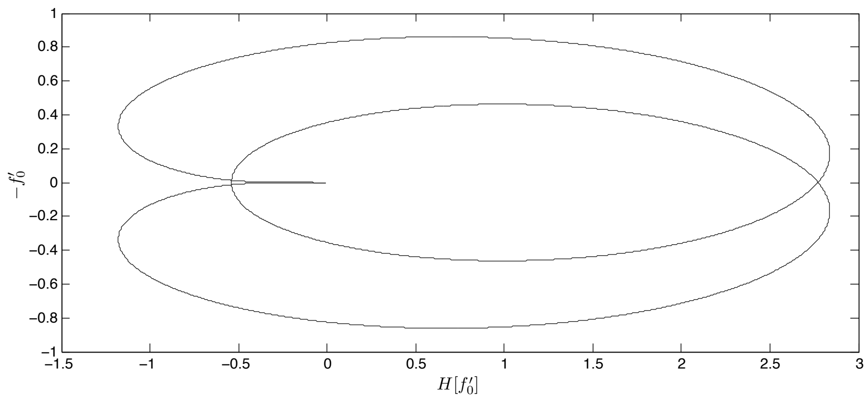
Figure 12.3. *Stable Penrose plot for a Maxwellian distribution function*

Penrose plots can be used to visually determine spectral stability. As described above, the Maxwellian distribution function is stable as the resulting ε-plot does not encircle the origin. However, it is not difficult to construct unstable distribution functions. In particular, the superposition of two displaced Maxwellian distributions, $f_0 = e^{-(p+c)^2} + e^{-(p-c)^2}$, is such a case. As c increases, the distribution goes from stable to unstable. This instability is known as the two-stream instability. Figures 12.4(a) and (b) demonstrate how the transition from stability to instability is manifested in a Penrose plot.

At the bifurcation point, the Penrose plot crosses the origin, indicating the vanishing of the dispersion relation on the real axis and therefore the presence of a member of the point spectrum. This eigenmode will be stable because $u \in \mathbb{R}$ and will be embedded within the continuous spectrum. Thus, the two-stream instability is an example of the CHH bifurcation.



(a)



(b)

Figure 12.4. *Penrose plots for a superposition of Maxwellian distribution functions with a) a stable separation and b) an unstable separation*

The description of the CHH bifurcation requires that we are able to assign an energy signature to the continuous spectrum. Because eigenfunctions associated with continuous spectra are not normalizable, this requires some delicacy. This was first done in the Vlasov context in [MOR 92], where a comparison to the usual energy signature for discrete modes was given, followed by a rigorous treatment of signature in [MOR 00]. In the context of shear flow, signature was defined in [BAL 01], in magnetofluids in [HIR 08] and for the general system described in the present section in [MOR 03]. A rigorous version of Kreĭn's theorem for the CHH bifurcation was given in [HAG 11b]. We will give a general description of this energy signature for the continuous spectrum in Chapter 13, but we motivate it here first by treating the analogous version of this instability in the context of the waterbag model, which will have the advantage of only possessing a discrete spectrum.

12.3.2.2. *Bifurcations in the waterbag model: Vlasov interpretation*

One important feature of the system [12.27] is that its solution is a symplectic rearrangement of the initial condition $\mathring{f}(q,p) = f(q,p,0)$, i.e. its solution has the form

$$f(q,p,t) = \mathring{f} \circ \mathring{z}(q,p,t), \qquad [12.36]$$

where $\mathring{z}(q,p,t) = (\mathring{q}(q,p,t), \mathring{p}(q,p,t))$ is a canonical transformation.

The rearrangement comes from the solution of the ordinary differential equation for a single particle in the self-consistent potential ϕ. This implies that the level set topology of the initial condition is preserved, which can be leveraged to simplify the equations in the case of certain types of initial conditions. One such simplification is known as the waterbag reduction (see, e.g., [BER 67]), in which it is assumed that the initial condition $\mathring{f}$ is a sum of characteristic functions. This property is preserved under composition with the symplectic map $\mathring{z}$ so that the solution remains a sum of characteristic functions. The equations simplify to equations for the locations of the contours separating different regions of constant f. Piecewise constant initial conditions lead to a fluid closure that is exact for waterbag initial conditions, and the $1 + 1$ theories in the previous section can be seen to arise from such an ansatz. We will exploit the reduction by using a layered waterbag or onion-like initial condition to closely approximate a continuous distribution function that undergoes the bifurcation to linear instability we are interested in. In this way, we will be able to connect the HH bifurcation with the CHH bifurcation that we describe later.

We begin by assuming f to be a piecewise constant between M curves $p_\alpha(q,t)$, i.e.

$$f(q,p,t) = f_\alpha \qquad \text{if} \qquad p_\alpha < p < p_{\alpha+1},$$

where f_α is a positive constant. The equations for the curves p_α come from the equations of single particle motion for a particle at $(p_\alpha(q,t), q)$,

$$p_{\alpha,t} + p_\alpha\, p_{\alpha,q} = -\phi_q \quad \text{and} \quad \phi_{qq} = 1 - \sum_\alpha f_\alpha (p_{\alpha+1} - p_\alpha),$$

and this system is Hamiltonian, with Hamiltonian function being the classical energy

$$H = \sum_\alpha \frac{\Delta f_\alpha}{6} \int_{\mathbb{T}} dq\, p_\alpha^3 + \frac{1}{2} \int_{\mathbb{T}} dx\, (\phi_q)^2 \ .$$

Here, $\Delta f_\alpha = f_{\alpha-1} - f_\alpha$, is the change in the distribution function when crossing the αth waterbag layer. The Poisson bracket is similar to those seen in Hamiltonian fluid theories [MOR 98]

$$\{F, G\} = \sum_\alpha \int_{\mathbb{T}} \frac{dq}{\Delta f_\alpha} \frac{\delta F}{\delta p_\alpha} \partial \frac{\delta G}{\delta p_\alpha}.$$

The equilibria of the waterbag model that we are interested in studying are charge neutral and spatially homogeneous, $p_\alpha = p_{\alpha 0}$ constant, such that the electric potential $\phi \equiv 0$. We chose such a state and linearize about it, yielding the equations of motion

$$p_{\alpha,t} + p_{\alpha 0}\, p_{\alpha,q} = -\phi_q \quad \text{and} \quad \phi_{qq} = -\sum_\alpha f_\alpha (p_{\alpha+1} - p_\alpha)\,.$$

Moving to Fourier space and eliminating the dependence on q in favor of the wave number k gives

$$p^\alpha_{k,t} + ik\, p^\alpha_0 p^\alpha_k = ik \sum_\alpha f_\alpha (p^{\alpha+1}_k - p^\alpha_k),$$

the equations of motion for the Fourier coefficients. In terms of the Fourier coefficients, the Hamiltonian of the linearized system is

$$H = -\frac{1}{2} \sum_{k\in\mathbb{Z}} \left(\sum_\alpha p^\alpha_0 \Delta f_\alpha\, |p^\alpha_k|^2 + k^2 |\phi_k|^2 \right) .$$

Here, the term $-p^\alpha_0 \Delta f_\alpha$ arises from the term $-pf_0'$ in the linearized Vlasov equation, which indicates the signature of the continuous spectrum. The bracket is the bracket of the original nonlinear system written in terms of the Fourier modes

$$[f, g] = \sum_{k\in\mathbb{N}} \sum_\alpha \frac{ik}{\Delta f_\alpha} \left(\frac{\partial f}{\partial p^\alpha_k} \frac{\partial g}{\partial p^\alpha_{-k}} - \frac{\partial g}{\partial p^\alpha_k} \frac{\partial f}{\partial p^\alpha_{-k}} \right) .$$

This bracket is non-degenerate, and therefore, the system is nearly canonical in terms of the new variables. In particular, for a given pair $k, -k$, the linear equations form a finite-dimensional canonical Hamiltonian system upon scaling similar to that of section 12.2.2.

The dispersion relation for this system, for a given wave number k, and $u = \omega/k$, is derived by multiplying the αth equation by Δf_α and summing, which is analogous to that for the Vlasov system,

$$\varepsilon(u,k) = 1 - \frac{1}{k^2}\sum_\alpha \frac{\Delta f_\alpha}{u - p_0^\alpha} = 0\,.$$

This dispersion relation can be analyzed graphically in terms of u. There are poles of the dispersion function where $u = p_0^\alpha$. For $u \in (p_0^\alpha, p_0^{\alpha-1})$, the dispersion function always has a zero if $\Delta f_{\alpha+1}$ has the same sign as Δf_α, because ϵ will converge to the opposite value of infinity at each end of the interval. Therefore, there will be at least one zero in each interval that has this property. In intervals where $\Delta f_{\alpha+1}$ and Δf_α have different signs, there are either no zeros or an even number of zeros, because ε must converge to the same value of infinity.

The reader may have noticed a similarity between the above formulas and the multifluid formulas of section 12.2.1. In fact, the waterbag models are examples of multifluid models, which are thus exact fluid closures of the Vlasov–Poisson system. This can be seen by writing the waterbag model in terms of new variables ρ_α and u_α given by

$$\rho_\alpha = \left(p_\alpha - p_{\alpha-1}\right)/f_\alpha \quad \text{and} \quad u_\alpha = \left(p_\alpha + p_{\alpha-1}\right)/2,$$

where ρ_α is a fluid density, and u_α is a fluid velocity. Under this change of variables, the equations governing the waterbag model take the following form:

$$\rho_{\alpha,t} + \left(u_\alpha \rho_\alpha\right)_q = 0, \quad \phi_{qq} = -\sum_\alpha \rho_\alpha$$
$$\left(\rho_\alpha u_\alpha\right)_t + \left(u_\alpha^2 \rho_\alpha + \rho_\alpha^3 f_\alpha^2/12\right)_q = -\rho_\alpha \phi_q\,.$$

Evidently, under the identification $p_\alpha = \rho_\alpha^3 f_\alpha^2/12$ (or $h_\alpha = \rho_\alpha^2 f_\alpha^2/8$), the above equations are identified as a multifluid Hamiltonian system.

The dispersion function can also be rewritten in terms of the new variables so that it resembles the analogous expressions (i.e. the diagravic or dielectric functions of the multifluid section). After linearizing around an equilibrium state with ρ_0^α, u_0^α, and then performing some algebraic manipulations yields

$$\varepsilon(k,\omega) = 1 - \sum_\alpha \frac{\rho_0^\alpha}{\left(\omega - k u_0^\alpha\right)^2 - k^2 (u_\theta^\alpha)^2}, \qquad [12.37]$$

where $u_\theta^\alpha := \sqrt{(\rho_0^\alpha)^2/f_\alpha^2}$ is a thermal velocity that measures the width in velocity space of a waterbag. Thus, bifurcations in the waterbag model, the Vlasov–Poisson system and Hamiltonian multifluid equations are all described using similar mathematical expressions.

Because the waterbag system is a finite-dimensional canonical linear Hamiltonian system, the standard results of that theory apply, including Kreĭn's theorem. We can therefore determine whether there are any unstable modes by counting the total number of neutral eigenvalues. If it is equal to the number of degrees of freedom of the system, then we can expect stability; otherwise, due to the fact that eigenvalues off the imaginary axis come in quartets, we can expect instability.

Now, we determine the signature of each of the stable discrete modes of the waterbag model. Beginning with the linearized equations, and assuming the normalization condition, $1 = \sum_\alpha \Delta f_\alpha \, p_\alpha/k = -k\phi_k$, we find the Fourier eigenvector $p_k^\alpha = 1/[k(p_0^\alpha - u)]$. Using this in the expression for the energy, we get a formula for the energy of a discrete mode, viz.

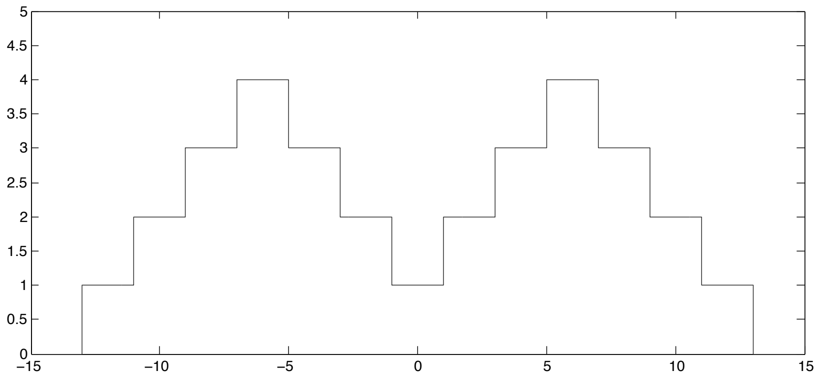
$$H = -\sum_\alpha \frac{p_0^\alpha}{2k^2} \frac{\Delta f_\alpha}{(p_0^\alpha - u)^2} + \frac{1}{2}\,. \qquad [12.38]$$

Next, replacing p_0^α in the numerator of [12.38] by $p_0^\alpha = u + (p_0^\alpha - u)$ leads to

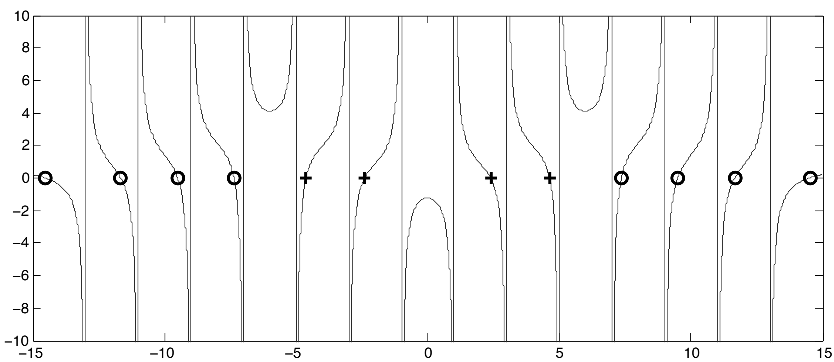
$$H = \frac{1}{k^2}\sum_\alpha \left(\frac{\Delta f_\alpha \, p_0^\alpha}{(u - p_0^\alpha)^2} + \frac{\Delta f_\alpha}{p_0^\alpha - u} \right) + \frac{1}{2} = \frac{1}{k^2}\sum_\alpha \frac{\Delta f_\alpha \, p_0^\alpha}{(u - p_0^\alpha)^2} = u\frac{\partial \varepsilon}{\partial u},$$

where in the last expression, we obtain the dielectric energy (with the electric field amplitude dependence scaled away).

The energy of a discrete mode is proportional to the derivative of the dispersion function at the frequency corresponding to the mode. As mentioned previously, this familiar formula is also true for embedded modes in the Vlasov equation [SHA 94], and is particularly convenient for use in the waterbag model because it allows geometric evaluation of the signature of modes in the waterbag model. Suppose at first that $u > 0$. Then, the signature of a mode is positive if the dispersion function is increasing at the mode and negative if it is decreasing at the mode. If Δf_α does not change sign from one interval to the next and there is one mode in the corresponding interval, the mode will have signature $-\,\mathrm{sgn}(p_0^\alpha \Delta f_\alpha)$. Similarly, any modes in the same interval must have opposite signature (or one must have zero signature) because the dispersion function must cross the axis in opposite directions at each discrete mode. An example of such a waterbag distribution function is plotted in Figure 12.5(a), and the dispersion relation is plotted in Figure 12.5(b), where we have marked the zeros with crosses and circles that indicate their signature. As noted, in the general case, there is exactly one mode in intervals where Δf_α does not change sign and either zero or two modes in intervals where it does change sign.



(a)



(b)

Figure 12.5. *a) Multilevel waterbag distribution. b) Plot of corresponding dispersion relation, with positive and negative signature modes indicated by circles and crosses, respectively*

Using the waterbag model, we can replicate the most important instabilities of the Vlasov–Poisson equation, in particular the two stream instability and bump on tail instability. Both of these instabilities can be emulated by a waterbag model with only a few "layers" (fluids). In particular, we will consider the special case of a waterbag with five layers as depicted in Figure 12.6. Observe that the outermost two have vanishing distribution function, i.e. $f_1 = f_5 = 0$, while we choose $f_2 = 1$, $f_3 = 0$ and $f_4 = 0.5$ so that the distribution has two peaks, one large and one small, separated by a valley. The stability of this model depends on the various parameters involved in defining the equilibrium. For a very large separation of the two peaks, the two-stream distribution function will be stable as depicted in Figure 12.7(a); as the peaks are moved closer together the two modes in the valley of the distribution function between the two peaks move closer together, eventually colliding, as depicted in Figure 12.7(b), and leaving the axis to become a pair of exponentially growing and decaying modes as depicted in Figure 12.7(c).

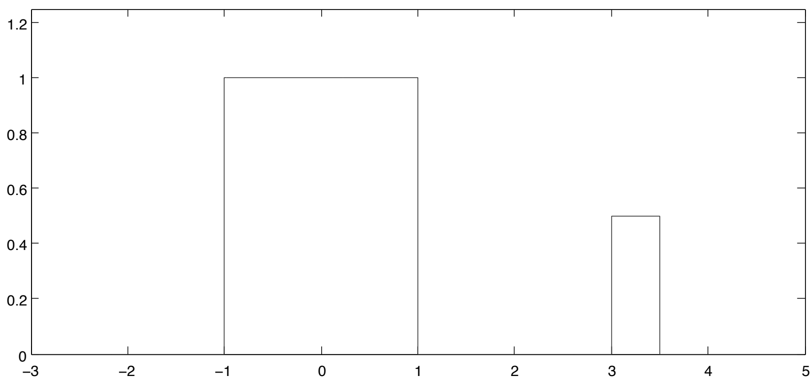
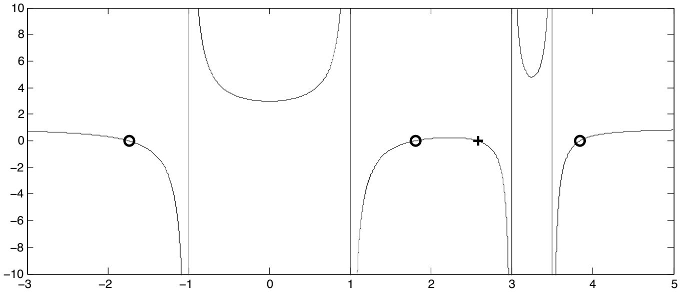
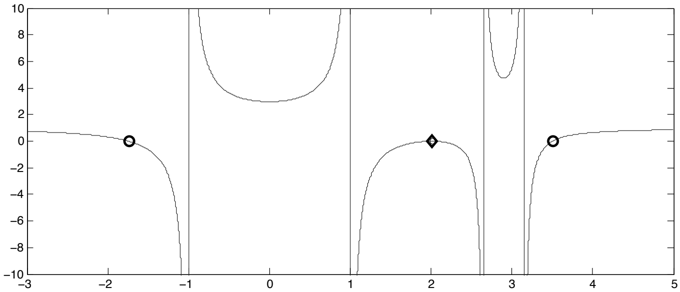


Figure 12.6. *Plot of a waterbag distribution function meant to capture the electron two-stream instability. As the small waterbag is moved closer to the large waterbag, a positive energy mode will collide with a negative energy mode and give rise to the two-stream or bump on tail instability*

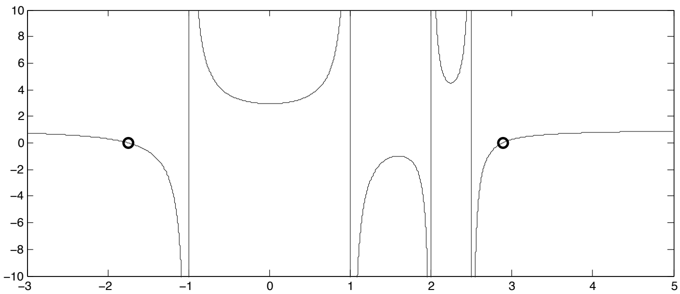
This transition here is identical to that which occurs in the two-stream instability of the Vlasov equation (or the corresponding bump on tail instability). In the waterbag case, there is a positive energy mode that collides with a negative energy mode in the valley of the distribution function.



(a)



(b)



(c)

Figure 12.7. *Dispersion function for a two-stream distribution function for parameter values corresponding to a) stable, b) neutral and c) unstable equilibria. Circles and crosses correspond to positive and negative energy modes, respectively, while the diamond indicates a mode at criticality.*

12.4. Summary and conclusions

In this chapter, we have described bifurcations in general classes of noncanonical Hamiltonian systems that describe, e.g., matter as fluid or kinetic theories. In the multifluid systems of section 12.2, we showed how to linearize, canonize and, for stable systems with discrete spectra, diagonalize to obtain a normal form. Hamiltonian bifurcations to instability were described, examples of SS bifurcations were given, but the emphasis was on the HH bifurcation. From the normal form, signature was identified, and it was seen that Kreĭn's theorem applies, just as for finite-dimensional systems. Next, the class of 2+1 Hamiltonian theories of section 12.3 were defined and considered. These theories generically posses continuous spectra when linearized, but the specific case of the Vlasov–Poisson systems was treated in detail. In particular, Penrose plots, which allow us to describe transitions to instability, via embedded modes in a continuous spectrum, were described. The technique here is of general utility, e.g. it was worked out also in detail for shear flow in [BAL 99]. It was also shown how to canonize the linearization of these 2+1 theories. Next, in order to understand the relationship between discrete bifurcations and the CSS and CHH bifurcations, we introduced the waterbag model, which is a reduction of the 2+1 class to a class of systems with a countable number of degrees of freedom, in which the continuous spectrum is discretized. The identification of the waterbag models with the multifluid models of section 12.2 was made and, consequently, the procedure for canonization and diagonalization of the waterbag models was established.

A main motivation for studying Hamiltonian systems is their universality, i.e. we are interested in understanding features of one system that apply to all. In this chapter we have shown how infinite-dimensional noncanonical Hamiltonian systems enlarge this universality class. It is clear that the same bifurcations occur in a variety of systems that describe different physical situations. Any specific system within our classes of systems may possess SS bifurcations, positive and negative energy modes and Kreĭn's theorem for HH bifurcations. Our aim is to show that an analogous situation transpires for CSS and CHH bifurcations. However, continuous spectra are harder to deal with mathematically and functional analysis is essential, but the existence of analogous behavior in the cases considered here guides us to develop a theory. For example, we can interpret the CHH bifurcation as an HH bifurcation with the second mode coming from the continuous spectrum. As stated before, the contents of this chapter are to set the stage for the explicit treatment of bifurcations with the continuous spectrum of Chapter 13, to which we direct the reader.

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