Chapter 11

Evolution Equations for Finite Amplitude Waves in Parallel Shear Flows

The evolution of a wave packet propagating on a shear flow with velocity profile $\bar{u}(y)$ can be investigated by using an amplitude $A(X, T)$ that varies slowly in space and time. In weakly nonlinear hydrodynamic stability problems, the partial differential equation (PDE) satisfied by A is often the Ginzburg–Landau equation. Whether or not this is the case depends on the dynamics of the critical layer, a thin layer centered on the point y_c, where $\bar{u} = c$, the perturbation phase speed.

Recently, alternative formulations have been employed, leading to an integro-differential equation governing the amplitude evolution. In such cases, it is the critical layer that dictates the form of the evolution equation. A finite amplitude approach that will be discussed in this chapter features dispersive effects dominant in the critical layer rather than viscosity. This approach is most appropriate in many geophysical shear flows, because the Reynolds numbers existent there are typically very large.

11.1. Introduction

The subject of hydrodynamic stability is linked with the evolution of a laminar flow toward a turbulent state. In the case of a parallel shear flow, the normal mode approach involves superimposing a perturbation of small amplitude on a shear flow $\bar{u}(y)$ in the x-direction. If the perturbation is assumed to be small, the governing equations can be linearized and a separation of variables is achieved by writing all perturbation quantities

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in the form $f(y)\ \exp\{i\,(kx - \omega t\}$. The wave number k is real, whereas $\omega = \omega_r + i\omega_i$ is complex. The foregoing procedure leads to an eigenvalue problem that determines $f(y)$ and, if we obtain $\omega_i > 0$ for certain values of the flow parameters, there is instability.

For most flows, linear theory yields a critical value of some parameter that, when exceeded, marks the onset of instability. Examples include the Rayleigh number for Bénard convection, the Taylor number for Taylor–Couette flow and the Reynolds number for Poiseuille flow. Near such a bifurcation point, the weakly nonlinear evolution of a disturbance can be investigated by expanding the velocity components in powers of the disturbance amplitude and $(R - R_c)$, where R_c denotes the critical value of the relevant parameter. This finite amplitude approach, pioneered by J.T. Stuart, has attracted a great deal of attention since the late 1950s. The basic idea is to replace the arbitrary constant multiplying the normal mode of linear theory by an amplitude function $A(t)$ satisfying the Stuart–Landau equation

$$\frac{1}{A}\frac{dA}{dt} = a_0 + a_2|A|^2 + O(|A|^4) + \cdots\ . \qquad [11.1]$$

In a frame of reference moving at the phase speed $c_r = \omega_r/k$, the constant a_0 can be identified with ω_i, the amplification factor of linearized theory (i.e. $\omega_i \sim (R - R_c)$ near R_c). However, it is the Landau constant a_{2r} that is of key importance from the viewpoint of nonlinear theory. Supposing for the sake of simplicity that a_2 is real, the following possibilities arise: (1) the case $a_2 < 0$ means that a linearly unstable perturbation $(a_0 > 0)$ will evolve toward a steady finite amplitude state having an equilibrium amplitude $|A_e|^2 = -(a_0/a_2)$. This is called the *supercritical* case. (2) If, on the other hand, it turns out that $a_2 > 0$, modes that would be damped $(a_0 < 0)$ according to linear theory can now amplify if their initial amplitude satisfies the condition $|A(0)|^2 > -(a_0/a_2)$. Such destabilization by finite perturbations is called *subcritical instability*.

The bifurcation approach following equation [11.1] has been most successful in flows such as Bénard convection and Taylor vortices, where instability is not observed experimentally until the critical Rayleigh or Taylor number is reached. These are examples of the supercritical case in which one state of flow loses stability to another state. A succession of bifurcations may occur to produce a complicated flow that is still not turbulent. This sequence of events contrasts greatly with plane Poiseuille flow, where turbulence was observed experimentally to occur at Reynolds numbers as low as 1,500, well below the linear critical value of 5,772.

A systematic finite amplitude approach for Poiseuille flow was published by Stuart in 1960, but it was not until 7 years later that Reynolds and Potter [REY 67] computed the Landau constant. With the mean flow normalized to be $\bar{u} = 1 - y^2$, it was found in [REY 67] that $a_2 = 31 - 173i$ so that $a_{2r} > 0$ and subcritical instability is possible. This was viewed at that time as highly encouraging, but two limiting factors must be kept in mind. First, the Hopf bifurcation occurring at $R_c = 5{,}772$ is far away from the lower values where transition can be observed. Second, if the initial amplitude $A(0)$ is large enough to cause subcritical instability, then more terms in the expansion [11.1] need to be computed. An authoritative review of the foregoing developments can be found in [STU 71].

A positive development, subsequent to the publication of [REY 67] and [STU 71], was that the experiments reported by Nishioka *et al.* [NIS 75] validated several aspects of both linear and finite amplitude theories. By controlling the background turbulence level, it was possible to maintain the laminar flow for Reynolds numbers as high as 9,000. Of most interest in the present context, the notion of subcritical instability was supported because a definite threshold amplitude was observed in data taken at a Reynolds number of 5,000.

Although the finite amplitude approach cannot predict transition to turbulence, it does offer some significant improvements compared with linear theory. In the supercritical case, an equilibrium amplitude is determined, whereas the linear prediction is that $A \sim \exp(a_0 t)$, implying that an unstable perturbation will amplify exponentially forever. Subcritical instability, on the other hand, is precluded in the linear theory because the initial perturbation amplitude is an arbitrary constant. Finally, it should be pointed out that following the onset of instability, perturbations exchange energy with the mean flow which is altered as a consequence. This effect can make a significant contribution to the Landau constant, but it is ignored in a linear analysis.

We conclude this section with an example from meteorology that leads, interestingly, to an amplitude equation that is second order in time. Changes in the mean flow play an important role in this example. Quasi-periodic variations in atmospheric variables with time scales of about a month have been of considerable interest for the last 60 years. They are important in weather forecasting and, without giving a formal definition, these aperiodic oscillations in tropospheric zonal flows are known as “vacillations” in the meteorological literature. They are believed to be a manifestation of baroclinic instability, and a detailed analysis of atmospheric data exhibiting this phenomenon can be found in [HUN 78].

A linear stability analysis, by itself, cannot explain the aforementioned vacillations because the only possibilities are exponential behavior. The case of interest is the supercritical one, but equation [11.1] predicts equilibration at a finite amplitude. The weakly nonlinear analysis of Pedlosky [PED 70], on the other hand, leads in the inviscid limit to the amplitude equation

$$\frac{d^2A}{dt^2} - A + NA(|A|^2 - |A(0)|^2) = 0, \qquad [11.2]$$

where the constant N is positive and a function of the zonal and meridional wave numbers. The nonlinear term $A|A|^2$ in [11.2] includes a contribution from the mean zonal flow, which oscillates as a result of its coupling to the baroclinic wave. When a weak effect of friction is included in the theory, a $\frac{dA}{dt}$ term is added to [11.2]. The more important fact is that the nonlinear term in [11.2] is coupled to an equation for the mean zonal flow. The reader is referred to [PED 87, section 7.16] for a thorough discussion of this problem and more recent extensions.

The model used in Pedlosky's analysis is a quasi-geostrophic, two-layer model with different velocities in each layer. To what extent the simplifications incorporated in this model affect the outcome is an open question. In the following section, we discuss how to anticipate the order of the amplitude equation that will result from a weakly nonlinear analysis. Finally, wave packet equations that are second order in time are given at the end of section 11.2.2.

11.2. Wave packets

The notion of a wave packet is a familiar one in mathematical physics and it is well known that the envelope of a packet propagates with the group velocity $\omega'(k)$. Benney and Newell [BEN 67] introduced a formulation using the method of multiple scales, which is well suited to study the weakly nonlinear propagation of waves in fluids. If $\mu << 1$ is a measure of modulations over a distance much larger than the wavelength, the amplitude is now allowed to vary slowly in space and time so that $A(t) \to A(X, T)$, where $X = \mu x$ and $T = \mu t$. In the multiple scaling method, derivatives in the original PDEs are transformed according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial T} \quad \text{and} \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial X}\,. \qquad [11.3]$$

11.2.1. *Conservative systems*

In order to treat weakly nonlinear problems, a dimensionless amplitude parameter ε is introduced, and for waves propagating in the x-direction the amplitude evolves according to

$$\mu \left(\frac{\partial A}{\partial T} + \omega' \frac{\partial A}{\partial X} \right) - \frac{1}{2}\, i\, \mu^2\, \omega''\, \frac{\partial^2 A}{\partial X^2} + \cdots = i\, \varepsilon^2 \gamma\, |A|^2\, A\,. \qquad [11.4]$$

The constant γ and the group velocity ω' in [11.4] are generally real for a conservative system. To obtain the desired balance between nonlinearity and dispersion, we set $\mu = \varepsilon$ and use a frame of reference moving at the group velocity. Introducing $\tau = \varepsilon^2 t$ and

$\xi = \varepsilon(x - \omega' t)$ as independent variables, the lowest order amplitude equation is as follows:

$$\frac{\partial A}{\partial \tau} - i\,\delta\,\frac{\partial^2 A}{\partial \xi^2} = i\gamma\,|A|^2\,A\,, \qquad [11.5]$$

where $\delta = \omega''/2$. This equation, now known as the nonlinear Schrödinger equation, can be solved by the inverse scattering method when γ and δ are real. Here, however, we are more interested in its use to investigate the potential sideband instability of permanent waves.

Benney and Newell [BEN 67] noticed that a special solution of [11.5] is given by the nonlinear discrete wave

$$A^{(0)}(\tau) = A(0)\,e^{i\gamma|A(0)|^2\tau}. \qquad [11.6]$$

An example to which this observation can be applied is Stokes waves. In 1847, Stokes used a perturbation expansion to investigate the effects of nonlinearity on the propagation of water waves. Besides modifying the displacement of the surface compared with the linear result, Stokes showed that there would be an $O(\varepsilon^2)$ correction to the frequency of the wave. In a much cited paper, Benjamin and Feir [BRO 67] tried, unsuccessfully, to generate Stokes waves in the laboratory. They recognized that this was due to an instability, which is now called modulational instability or, simply, Benjamin–Feir instability.

The aforementioned instability was explained theoretically in [BRO 67] by formulating an analysis of a resonant triad consisting of a primary wave of frequency ω and two smaller sidebands of frequencies $\omega(1 \pm \delta)$. This approach was successful in predicting the initial instability, even though the resonance conditions are not satisfied exactly. Four waves are required to satisfy the resonance conditions for water waves, but the primary wave can be input twice. An alternate approach, based on the nonlinear Schrödinger equation, is outlined below. This method is superior in that a wave packet description is closer to reality than a primary mode with two sideband perturbations.

Following [BEN 67], we examine the stability of the solution [11.6] by superimposing a perturbation as follows:

$$A(\xi,\tau) = A^{(0)}(\tau) + B(\xi,\tau)\,e^{i\gamma|A(0)|^2\tau}\,. \qquad [11.7]$$

Substituting [11.7] into [11.6] and linearizing, we find after differentiating once with respect to τ and some manipulation that $B(\xi,\tau)$ satisfies the PDE

$$\frac{\partial^2 B}{\partial \tau^2} + \delta^2 \frac{\partial^4 B}{\partial \xi^4} + 2\,\gamma\,\delta\,|A(0)|^2 \frac{\partial^2 B}{\partial \xi^2} = 0\,. \qquad [11.8]$$

Restricting our attention now to sideband perturbations, we write $B(\xi,\tau)$ as

$$B(\xi,\tau) = b(\tau)e^{i\kappa\xi}, \qquad [11.9]$$

which leads to the following second-order ordinary differential equation (ODE) for $b(\tau)$:

$$b'' + \delta\,\kappa^2(\delta\,\kappa^2 - 2\,\gamma\,|A(0)|^2)\,b = 0\,. \qquad [11.10]$$

It is easily seen that the finite amplitude discrete mode is stable if $\gamma\delta < 0$, but the sideband perturbation will grow exponentially if $\gamma\delta > 0$ and $\kappa^2 < 2\,\gamma\,|A(0)|^2/\delta$.

For the specific case of a Stokes wave on deep water, the linear dispersion relation is $\omega_0^2 = gk$, where g is the gravitational constant and it turns out that the condition leading to sideband instability is $\kappa^2 < 32\,k^2\,|A(0)|^2/|\omega_0|$. The prediction of instability should not be interpreted as meaning that chaotic motion will ensue because it is based on a linear analysis. What actually occurs is a modulation of the wave train in which energy is exchanged between the primary mode and the sidebands. Envelope solitons have been observed in the experiments of Yuen and Lake [YUE 80] and the reader is referred to their review article for details of the experiments. Extensions of the theory to include a finite depth and oblique waves are also discussed in [YUE 80].

An interesting historical review of the different approaches to the modulation instability, including applications in fields such as electromagnetics, has been given by Zakharov and Ostrovsky [ZAK 09]. Finally, we note that damping can enhance the Benjamin–Feir instability, which has been studied by Kirillov [KIR 13] by means of the $\mathcal{PT}$-symmetric method from quantum mechanics.

11.2.2. *Applications to hydrodynamic stability*

It was realized some years after the publication of [BEN 67] that equation [11.4] applies to problems in hydrodynamic stability, as well as conservative systems, the primary difference being that ω', ω'' and γ are now generally complex. This observation is shown below to be useful in deciding the order of the amplitude equation that will be obtained in the weakly nonlinear theories discussed in section 11.1. Let us use the example of the incompressible mixing layer $\bar{u} = \tanh y$ to illustrate this.

The mixing layer differs from the examples discussed in connection with equation [11.1] of section 11.1 in that there is no critical Reynolds number. The reason is that the mechanism of instability is inviscid and associated with the inflection point in the velocity profile at $y = 0$. This point is a vorticity maximum; so Fjørtoft's necessary condition for instability is satisfied (see [DRA 81] for an exposition of stability theory). The neutral solution is known in closed form; if $\phi(y)$ is the eigenfunction for the stream function, the neutral mode is $\phi = \operatorname{sech} y, c = 0$ and $k = 1$. Using the weakly nonlinear wave packet equation [11.4], Benney and Maslowe [BEN 75] derived the amplitude equation

$$\frac{\partial A}{\partial T} - \frac{2i}{\pi}\frac{\partial A}{\partial X} = -\frac{16}{3\,\pi}\,|A|^2\,A\,, \qquad [11.11]$$

where the balance $\mu = \varepsilon^2$ was used.

The Landau constant $a_2 = -16/3\pi$ was derived earlier by Schade [SCH 64], which was the first application of weakly nonlinear theory to a shear flow. To recover Schade's result, we note that Lin's perturbation formula for perturbing away from a neutral solution yields $\omega_i \simeq \frac{2}{\pi}\,(1-k)$ for the $\tanh y$ shear layer. Next, we separate variables by writing $A(X,T) = a(\varepsilon^2 t)\exp(-\frac{1}{2}i\pi X)$ to obtain

$$\frac{1}{a}\frac{da}{dt} = \omega_i - \varepsilon^2\,\frac{16}{3\,\pi}\,a^2, \qquad [11.12]$$

which is in the form of the amplitude equation [11.1].

Given that all the coefficients of the linear terms in [11.4] involve derivatives of $\omega(k)$, having an analytical expression for the dispersion relation is clearly advantageous. We consider, therefore, the weakly nonlinear development for the three-layer approximation to the $\tanh y$ mixing layer, namely

$$\bar{u}(y) = \begin{cases} \operatorname{sgn}(y) & |y| > 1 \\ y & |y| \le 1\,. \end{cases}$$

The dispersion relation for this three-layer model was found by Rayleigh and it is as follows:

$$\omega^2 = (1 + 4k^2 - 4k - e^{-4k})/4\,. \qquad [11.13]$$

According to [11.13], the neutral wave number $k_n \approx 0.64$ compared with $k_n = 1$ for the $\tanh y$ mixing layer and instability occurs in both cases for $k < k_n$. The maximum amplification rate is $\omega_i \approx 0.20$ at $k = 0.40$ and this is very close to the corresponding result for the continuous profile, namely $\omega_i \approx 0.19$ at $k = 0.44$. Clearly, the broken-line model is very useful in the context of linear stability theory. That this is not the case for weakly nonlinear theory can be readily deduced from the amplitude evolution equation [11.4], as we now show.

The key to anticipating the form of the amplitude equation is the behavior of the complex group velocity as the stability boundary is approached from the unstable side. Differentiating [11.13], we obtain

$$\omega'(k) = i\,\frac{1 - 2k - e^{-4k}}{[4k + e^{-4k} - 1 - 4k^2]^{1/2}}\,. \qquad [11.14]$$

As $k \uparrow 0.64$, the denominator of [11.14] vanishes so that ω_i' becomes infinite, unlike the continuous case, where from [11.11], we see that $\omega_i' = -2/\pi$. On the other hand, the broken-line profile does give the correct value for the real part of ω', as it vanishes for both the continuous and the three-layer model.

When $\omega'(k) \to \infty$ at the bifurcation point marking the onset of instability, the amplitude equation will generally be second order in time. The reasoning leading to that conclusion follows from the wave packet formulation on which [11.4] is based. Specifically, the evolution equation [11.4] was viewed as an expansion of $\partial A/\partial T$. In the exceptional case, $\omega'(k) \to \infty$, we can think of the amplitude equation as being an expansion of $\partial A/\partial X$ having the form

$$\frac{\partial A}{\partial X} = \frac{1}{\omega'}\frac{\partial A}{\partial T} - \frac{i}{\beta}\frac{\partial^2 A}{\partial T^2} + \cdots\,,$$

where the constant β is determined from a solvability condition imposed at higher order. The details for the broken-line mixing layer model have been worked out by Benney and Maslowe [BEN 75] who obtained the amplitude equation

$$\frac{\partial^2 A}{\partial T^2} - 0.201\,i\,\frac{\partial A}{\partial X} = -0.424\,|A|^2\,A\,.$$

To conclude this section, we should note that an amplitude equation that is second order in time does not necessarily mean that the group velocity is infinite. This was demonstrated by means of a quite general formulation for wave packets by Weissman [WEI 79]. The specific flow to which the analysis was applied was a Kelvin–Helmholtz flow, i.e. a vortex sheet between two fluids of different density with surface tension included. The dispersion relation in [WEI 79] was written implicitly in the form $F(\omega, k, U)$, where U is the velocity jump across the interface and two slow variables were used in both space and time. The amplitude equation for the particular flow investigated turned out to be

$$\frac{\partial^2 A}{\partial T^2} - \omega_k^2 \frac{\partial^2 A}{\partial X^2} = A + N\,|A|^2\,A\,,$$

where ω_k^2 is the group velocity and it is real at the critical value of U.

11.2.3. *The Ginzburg–Landau equation*

The finite amplitude approach discussed in section 11.1 can be readily generalized to treat wave packets using the multiple scaling technique. The details are quite similar to the derivation of equation [11.5] in section 11.2.1, the nonlinear Schrödinger equation. There will, however, be two differences in the resulting amplitude equation: first, a linear term must be present, allowing the amplification or decay of the perturbation and, second, the coefficients of the dispersive and nonlinear terms may be complex. The resulting PDE is often called the Ginzburg–Landau equation, which can be written as

$$\frac{\partial A}{\partial \tau} - i\,\delta\,\frac{\partial^2 A}{\partial \xi^2} = \left(\frac{a_0}{\varepsilon^2}\right)\,A + a_2\,|A|^2\,A\,. \qquad [11.15]$$

Exactly as in [11.5], the coefficient $\delta = \omega''/2$ and the independent variables are $\tau = \varepsilon^2 t$ and $\xi = \varepsilon(x - \omega' t)$. The $O(\varepsilon^2)$ constant $a_0 \sim (R - R_c)$, as in section 11.1, and a_2 is again the Landau constant.

Equation [11.15] was first derived in hydrodynamic stability for the Bénard convection problem and not long after for Poiseuille flow. One difference is that the group velocity is zero for Bénard convection, as well as in the Taylor vortex problem; so it is obvious that there will be no $\frac{\partial A}{\partial \xi}$ term. The derivation of [11.15] for plane Poiseuille flow by Stewartson and Stuart [STE 71] is less evident and it involves some interesting mathematics. In the supercritical case, they solved the linear initial value problem, including viscosity. Using the method of steepest descent, they showed that the solution as $t \to \infty$ involves the independent variable $\xi = \varepsilon(x - \omega_k t)$ in a natural way and the asymptotic solution provides the initial condition as $\tau \to 0$ for [11.15]. It is advantageous to follow the fastest growing wave, because $\partial\omega_i/\partial k$ then vanishes and the group velocity is real.

To motivate the next section, a few words should be said about the procedure used in [STE 71] to determine the constants δ and a_2. The eigenvalue problem is solved at the lowest order in ε and higher order terms satisfy non-homogeneous ODEs. Integrating these ODEs between the boundaries leads to a solvability condition that determines the desired constants. Singularities do not arise because viscosity is present at all orders. However, in many applications, a primarily inviscid approach is more pertinent. In such cases, other effects must be introduced in place of viscosity to deal with the singularities that occur at critical points. The way in which these singularities are treated determines the magnitude of the constants and even the form of the amplitude equation. We now proceed to a description of these more recent methods.

11.3. Critical layer theory

We begin with a review of those aspects of the classical theory directly related to the focus of this chapter, namely amplitude evolution equations. Supposing the flow is two-dimensional and incompressible, it is convenient to use a stream function $\psi(x, y)$ related to the horizontal and vertical velocity components by $(u, v) = (\psi_y, -\psi_x)$. The basic equation describing the evolution of the flow is the vorticity equation, which can be written as

$$\zeta_t + \psi_y\,\zeta_x - \psi_x\,\zeta_y = R^{-1}\,\nabla^2\zeta, \qquad [11.16]$$

where the vorticity $\zeta = -\nabla^2\psi$ and R is the Reynolds number.

The mean and fluctuating parts of the stream function are now separated by writing $\psi(x, y, t) = \bar{\psi}(y) + \varepsilon\hat{\psi}(x, y, t)$, with $\varepsilon << 1$. Substituting for ψ in [11.16] leads to the PDE determining the evolution of the perturbation $\hat{\psi}$, namely

$$\hat{\zeta}_t + \bar{u}\,\hat{\zeta}_x + \bar{u}''\hat{\psi}_x + \varepsilon(\hat{\psi}_y\,\hat{\zeta}_x - \hat{\psi}_x\,\hat{\zeta}_y) = R^{-1}\,\nabla^2\hat{\zeta}\,, \qquad [11.17]$$

where $\hat{\zeta} = -\nabla^2\hat{\psi}$. If we set $\varepsilon = 0$ in [11.17], the PDE governing linear theory is obtained. Separating variables by writing $\hat{\psi} = \phi(y)\exp\{ik(x - ct)\}$ finally leads to the equation for $\phi(y)$, namely the Orr–Sommerfeld equation

$$(\bar{u} - c)(\phi'' - k^2\phi) - \bar{u}''\phi = \frac{1}{ikRe}\,(\phi^{iv} - 2k^2\phi'' + k^4\phi)\;. \qquad [11.18]$$

In the temporal theory, the wave number k is real, c is complex and kc_i is the amplification factor of an unstable perturbation. On a solid boundary, both ϕ and ϕ' must vanish, whereas exponential decay is usually imposed as a boundary condition if the flow is unbounded on one side or both sides.

Suppose that the Reynolds number R is large, as it is in most important applications. As an approximation, the right-hand side of [11.18] can be neglected, thereby leading us to Rayleigh's equation. According to Howard's semi-circle theorem [DRA 81], $(\bar{u} - c)$ must vanish somewhere for any neutral mode that lies on a stability boundary, i.e. a mode that is the limit of an unstable perturbation as $c_i \to 0$. Such a mode is not necessarily singular because it often happens that $\bar{u}'' = 0$ at the same value of y where $\bar{u} = c$. We have already seen an important example, the tanh y mixing layer discussed in section 11.2.2.

This is an appropriate place, however, to make the point that in finite amplitude theories there is no way of escaping the critical point singularity. In an inviscid, weakly nonlinear development, $\hat{\psi}$ is expanded in powers of ε and higher order terms satisfy non-homogeneous ODEs that will have terms multiplied by $(\bar{u} - c)^{-1}$ on the right-hand side. In the weakly nonlinear development leading to [11.12], Schade [SCH 64] encountered this difficulty at $O(\varepsilon^2)$ in the expansion of $\hat{\psi}$. This is the order that determines the Landau constant and a result that follows from viscous critical layer theory was invoked in [SCH 64] to determine a_2.

For the tanh y mixing layer, and for other velocity profiles with inflection points, viscosity can be introduced to deal with any singularity that appears at higher order. But the subject of critical layers first arose in the context of the Orr–Sommerfeld equation for flows with no inflection point. Taking Poiseuille flow again as an example, according to Rayleigh's inflection point criterion, the flow must be stable. Let us suppose that viscosity can somehow have a destabilizing effect and that [11.18] admits unstable solutions. Moreover, guided by Howard's semi-circle theorem, we suppose that for such modes, $\bar{u} = c$ at one or more interior points. (If c is complex, this singularity will be in the complex plane. But c_i is usually small and the singularity is then close enough to the real axis to be of concern.) Even for $R >> 1$, we will assume that the singularity is telling us that viscosity is significant in a thin "critical layer" centered on the point y_c, where $\bar{u} = c$, and this leads us to the viscous critical layer theory.

11.3.1. *Asymptotic theory of the Orr–Sommerfeld equation*

To begin, we suppose that the solution of [11.18] can be expressed as a series in powers of $\delta = (kR)^{-1}$. The lowest order term in the expansion, $\phi^{(0)}$, satisfies the Rayleigh equation and it provides an adequate representation of the solution everywhere except near a solid boundary or at a critical point y_c. The method of Frobenius can be used to express the solution of $\phi^{(0)}$ in the neighborhood of y_c as a linear combination of two power series

$$\phi_A = (y-y_c)+\frac{\bar{u}''_c}{2\bar{u}'_c}(y-y_c)^2+\cdots \quad \text{and} \quad \phi_B = 1+\cdots+\frac{\bar{u}''_c}{\bar{u}'_c}\,\phi_A\,\log(y-y_c)+\cdots. \quad [11.19]$$

The logarithmic singularity in ϕ_B leads to two difficulties in the case of a neutral or nearly neutral mode. First, the horizontal perturbation velocity is proportional to ϕ', which becomes unbounded as $y \to y_c$. Second, the eigenvalue problem for Rayleigh's equation cannot be solved until how to write the log term in ϕ_B when $y < y_c$ has been decided. To decide these matters, a thin critical layer must be added in which at least one viscous term enters the primary balance.

The Orr–Sommerfeld theory has been presented elsewhere in great detail (see [DRA 81], for example). Here, only the details essential to what follows will be given. To begin, we introduce the "inner variables" χ and η defined by

$$\eta = \frac{y - y_c}{\delta^{1/3}} \quad \text{and} \quad \chi(\eta) = \phi(y)\,, \quad \text{where} \;\; \delta = (kR)^{-1}. \qquad [11.20]$$

Substitution of these variables into [11.18] leads us to the critical layer equation

$$\chi^{iv} - i\bar{u}_c'\eta\chi'' = \delta^{1/3} i\bar{u}_c''(\eta^2\chi''/2 - \chi) + O(\delta^{2/3})\,. \qquad [11.21]$$

If χ is expanded in powers of $\delta^{1/3}$, it turns out that the first two terms are required in order to determine the correct branch of the logarithm to take below y_c. It can be seen that the left-hand side of [11.21] is Airy's equation for χ''; so the solution involves integrals of the Airy function (or Hankel functions of order 1/3).

The details are challenging, but expanding the solution for large $|\eta|$ and matching to [11.19] lead to the conclusion that for $y < y_c$ and $\bar{u}_c' > 0$, we must write $\log(y - y_c) = \log|y - y_c| - i\,\pi$. One says, in that case, that there is a "$-\pi$ phase change" across the critical layer. This causes a jump in the Reynolds stress $\tau \equiv -\overline{u'v'}$ that leads to the well-known Tollmien–Schlichting mechanism of instability. The Tollmien–Schlichting modes were predicted by the theory years before actually being observed in boundary layer experiments.

11.3.2. *Nonlinear critical layers*

The approach outlined in this section recognizes that nonlinearity may be more important in the critical layer than it is elsewhere. We recall that the Orr–Sommerfeld theory follows from equation [11.17] by essentially taking the limits $\varepsilon \to 0, R \to \infty$, in that order. If we re-examine [11.17], it can be seen that even if viscosity is neglected (i.e. $R \to \infty$), the vorticity equation is not singular provided that the nonlinear terms are retained.

An asymptotic approach based on this observation was first formulated by Benney and Bergeron [BEN 69]. Using matched asymptotic expansions, it was shown in [BEN 69] that a balance between linear and nonlinear inertial terms could be achieved by using a critical layer of thickness $O(\varepsilon^{1/2})$. Restricting our attention now to neutral modes, it is convenient to introduce a total stream function

$$\psi = \int_{y_c}^{y} (\bar{u} - c)\, dy + \varepsilon\, \hat{\psi}(\xi, y), \qquad [11.22]$$

where c is the phase speed, $\xi = kx$ and the flow is steady in a coordinate system traveling at speed c. Expanding $(\bar{u} - c)$ in a Taylor series near y_c and noting that according to [11.19], $\hat{\psi} \sim O(1)$ as $y \to y_c$, we see that the mean flow and perturbation are both $O(\varepsilon)$. It is, therefore, appropriate to define inner variables Y and Ψ as follows:

$$y - y_c = \varepsilon^{1/2} Y \quad \text{and} \quad \psi(\xi, y) = \varepsilon\, \bar{u}_c' \Psi(\xi, Y)\,. \qquad [11.23]$$

Introducing these variables in the vorticity equation [11.16], the governing equation in the critical layer takes the form

$$\Psi_Y \Psi_{YY\xi} - \Psi_\xi \Psi_{YYY} + O(\varepsilon) = \lambda\, \Psi_{YYYY}, \qquad [11.24]$$

where $\lambda \equiv 1/(k\, R\, \varepsilon^{3/2})$. The parameter λ is seen to be a measurement of the ratio of two critical layer thicknesses, i.e. $\lambda^{1/3} = \delta_{visc}/\delta_{NL}$, where, according to [11.20], $\delta_{visc} = (kR)^{-1/3}$ and we are interested here in the case $\lambda \ll 1$.

Although the details of the nonlinear critical layer theory are too involved for presentation here, we can still outline the analysis and state the most significant results. First, we observe that to the lowest order in ε, the solution of [11.24] satisfying the matching condition of the outer expansion is simply

$$\Psi^{(0)} = \frac{Y^2}{2} + \cos\xi. \qquad [11.25]$$

It is noteworthy that [11.25] is a solution of [11.24] even when $\lambda \sim O(1)$, i.e. when both viscosity and nonlinearity are present. The streamline pattern associated with [11.25] is known as Kelvin's cat's eyes and is illustrated in Figure 11.1.

The phase change across the critical layer is determined at $O(\varepsilon^{1/2})$ by matching the outer solution to $\Psi^{(1/2)}$, the $O(\varepsilon^{1/2})$ term in the expansion of Ψ. This can be seen by writing the log term in [11.19] as $\log|y - y_c| + i\,\theta_R$ for $y < y_c$, where θ_R is called the phase change. Although the PDE satisfied by $\Psi^{(1/2)}$ is linear, finding a solution continuous throughout the critical layer proves to be a formidable task. First, all harmonics of the fundamental perturbation become the same order of magnitude.

Solutions outside of the closed streamline region can be found as integrals, but these cannot be matched to the solution inside where, according to the Prandtl–Batchelor theorem, the vorticity must be a constant. To smooth out discontinuities in vorticity along the critical streamline $\Psi^{(0)} = 1$, viscous shear layers of thickness $O(\lambda^{1/2})$ must be included, as indicated in Figure 11.1.

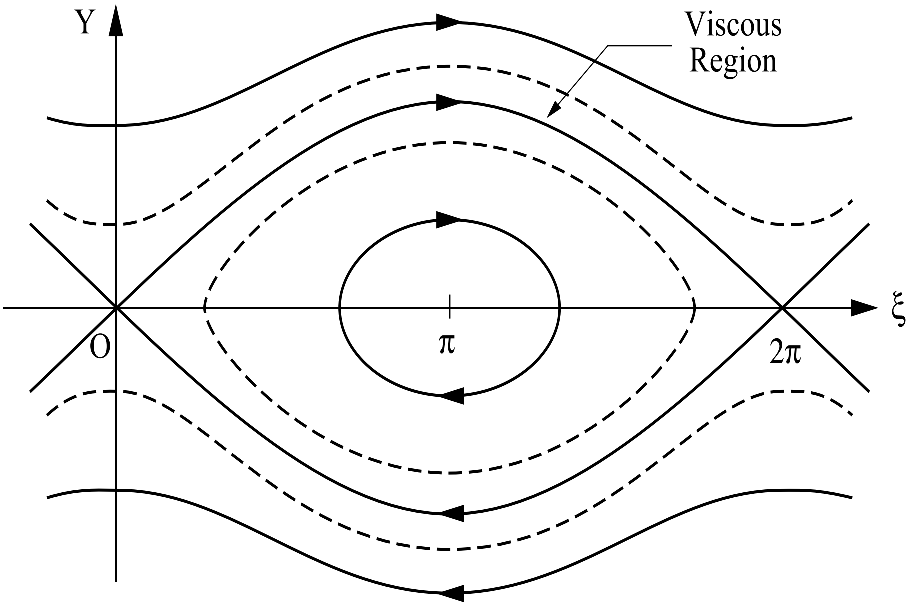

Figure 11.1. *Streamline pattern in the nonlinear critical layer*

Once a solution having both continuous vorticity and velocity has been found, matching to the linear, inviscid outer flow leads to the conclusion that the only solutions compatible with a nonlinear critical layer must have *zero* phase change. As a result, new solutions to the Rayleigh equation exist and these were computed for various flows in [BEN 69]. These neutral mode solutions can often be found in regions of parameter space where linear modes would be damped. This property may make them especially pertinent in geophysical fluid dynamics.

The nonlinear critical layer idea was developed initially for wave motions that are steady in a frame of reference moving with the wave. These nonlinear modes do not have to be near stability boundaries and the objective of the wave packet formulation of Benney and Maslowe [BEN 75] was to generalize the theory to allow slow amplitude variation in space and time. A particularly novel extension of these ideas was the application to solitary Rossby waves by Redekopp [RED 77]. For long waves, $\omega'' = 0$ so that the amplitude equation [11.4] becomes of Korteweg-de Vries type, the nonlinear term being $A\,A_\xi$ or A^2A_ξ; in the first case, density stratification is neglected in the vertical. It was suggested in [RED 77] that the theory offers an explanation for the Great Red Spot in Jupiter's atmosphere. In the barotropic (constant density) case, Caillol and Grimshaw [CAI 07] showed that mean flow distortions in the critical layer, neglected in [RED 77], lead to an additional nonlinear term in the Korteweg-de Vries equation and a different scaling for ξ and τ.

11.3.3. *The wave packet critical layer*

It has been pointed out many times [DRA 81] that the critical point singularity occurs as a result of considering a single normal mode, and that a superposition of many modes would be required to describe an arbitrary initial perturbation. The most general approach to the linear initial-value problem is to take a Fourier transform in x and a Laplace transform in time. That approach is technically difficult and, even in the simplest problems, incorrect results have been published (some of these are cited and corrected in [BRO 80]). In this section, a new, i.e. relatively recent, approach to deal with critical layers in shear flows is described. This method, developed in [MAS 94], is considerably less complex than doing the initial-value problem while retaining its most important features.

The theory outlined in this section is intended, primarily, for applications in geophysical fluid dynamics, where Reynolds numbers are typically very large. Therefore, we begin by setting $\varepsilon = R^{-1} = 0$ in [11.17]. The linear disturbance equation can then be written as

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\hat{\psi} - \bar{u}''\,\frac{\partial\hat{\psi}}{\partial x} \;=\; 0\,. \qquad [11.26]$$

Given that we are dealing with linear wave packets, the approach of section [11.2] will be followed and $\hat{\psi}$ can be written as

$$\hat{\psi} = \phi(X, y, T)e^{ik(x-ct)}\,.$$

Using the multiple scaling transformation [11.3], we find that ϕ satisfies the PDE

$$\left[(\bar{u}-c) - \frac{i\mu}{k}\left(\frac{\partial}{\partial T} + \bar{u}\frac{\partial}{\partial X}\right)\right]\left[\frac{\partial^2\phi}{\partial y^2} - k^2\left(1 - \frac{i\mu}{k}\,\frac{\partial}{\partial X}\right)^2\phi\right] - \bar{u}''\left(1 - \frac{i\mu}{k}\,\frac{\partial}{\partial X}\right)\phi = 0\,. \qquad [11.27]$$

It can be seen that when $\mu = 0$, Rayleigh's equation is recovered. We seek a solution of [11.27] by expanding ϕ in powers of μ and separate variables by writing

$$\phi = A(X,T)\,\phi_1(y) + \mu A_X\,\phi_2 + \mu^2 A_{XX}\,\phi_3 + \cdots\,, \qquad [11.28]$$

where A satisfies the usual equation for a linear wave packet, namely

$$\frac{\partial A}{\partial T} + \omega'\frac{\partial A}{\partial X} - \frac{1}{2}\,i\,\mu^2\,\omega''\,\frac{\partial^2 A}{\partial X^2} + \cdots = 0\,. \qquad [11.29]$$

In the absence of singularities, the coefficients ω' and ω'' would be evaluated by imposing solvability conditions on the non-homogeneous ODEs satisfied by ϕ_2 and ϕ_3 (see section 2 of [BEN 75] for details). However, the integrals resulting from imposing the Fredholm alternative are singular at the critical point; so the procedure is modified by introducing a wave packet critical layer. The series [11.28] now becomes the outer expansion and the continuation across y_c of the ϕ_i, all of which are singular, will be determined by matching to the critical layer solution.

One detail that should be fixed before moving to the critical layer formulation is the normalization of ϕ_1, which satisfies Rayleigh's equation. In terms of the Frobenius expansions [11.19], we can write $\phi_1 = a\phi_A + b\phi_B$ and choose $b = 1$ as the arbitrary constant in the eigenvalue problem. The constant a will be different above and below the critical layer if there is a phase change. Corresponding to this normalization, the behavior of ϕ_2, for example, is

$$\phi_2 \sim i\,\frac{\bar{u}_c''(c_g - c))}{k\bar{u}_c'^2}\,\log(y - y_c) + \cdots\,, \qquad [11.30]$$

where, of course, $c_g = \omega'$, the group velocity.

We begin an outline of the critical layer analysis by observing that [11.27] is singular only if $\mu = 0$. A critical layer of thickness μ can, therefore, be used, whose inner variables are defined by

$$y - y_c = \mu Y \quad \text{and} \quad \phi(X,y,T) = \Phi(X,Y,T)\,. \qquad [11.31]$$

Expanding the variable coefficients in [11.27] in the Taylor series and introducing the inner variables [11.31], we find that the PDE satisfied by Φ is as follows:

$$\left[\bar{u}_c'\, i\, k\, Y + (c - c_g)\,\frac{\partial}{\partial X} + \frac{i\mu}{k}\left(\frac{1}{2}\bar{u}_c'' k^2 Y^2 - \bar{u}_c'\, i\, k\, Y\frac{\partial}{\partial X} - d\frac{\partial^2}{\partial X^2}\right)\right] \times \Phi_{YY}$$

$$-i\, k\, \mu\, \bar{u}_c''\Phi + O(\mu^2) = 0\,. \qquad [11.32]$$

The behavior of the outer expansion for small $(y - y_c)$ determines the form of the inner expansion that can be matched to it. This form turns out to be

$$\Phi \sim \Phi^{(0)} + \bar{\mu}\log\bar{\mu}\,\Phi^{(1)} + \bar{\mu}\,\Phi^{(2)} + \cdots\,, \qquad [11.33]$$

where $\bar{\mu} = i\mu/k$ and the term that will determine the phase change is $\Phi^{(2)}$. At the lowest order, the matching condition is $\Phi^{(0)} \sim A(X,T)$ as $Y \to \infty$ and it can be seen easily that this is also a solution of [11.32] when $\mu = 0$. It is, in fact, the only acceptable solution because the general solution contains rapid oscillations that will not permit matching to ϕ.

In the same way, $\Phi^{(1)}$ is simply equal to its asymptotic behavior for large Y. The important term, as indicated above, is $\Phi^{(2)}$ and it must satisfy

$$\left[\bar{u}_c'\, i\, k\, Y + (c - c_g)\,\frac{\partial}{\partial X}\right]\,\Phi^{(2)}_{YY} = k^2\,\bar{u}_c''\,\Phi^{(0)} = k^2\,\bar{u}_c''\,A. \qquad [11.34]$$

Equation [11.34] was solved in two different ways in [MAS 94]. This was important because by extending the theory to stratified shear flows only one of these could be used, which was to use a Fourier transform in X. The fact that a direct solution of [11.34] can also be obtained is important in order to verify that the results of the two methods are the same. Without giving all the details, let us summarize the new features in each method.

Beginning with the direct method, a straightforward integration by parts permits [11.34] to be integrated once with respect to Y and, additional integrations with respect to X and Y, lead to the general solution

$$\Phi^{(2)} = \Phi_h^{(2)} + \frac{i\,k\,\bar{u}_c''}{(c - c_g)}\int_0^Y (Y - Y')\,dY' \int_\infty^X A(X',T)e^{-ip(X'-X)Y'}dX'. \qquad [11.35]$$

The homogeneous solution $\Phi_h^{(2)} = B(X,T) + D(X,T)Y$ and $p = k\bar{u}_c'/(c_g - c)$. It is assumed here that c_g and c are real with $c_g > c$, so that far upstream ($X \to \infty$), the presence of the packet is not sensed. This is the correct condition because [11.29] implies that we are moving at the group velocity; so individual waves are not ahead of the packet. In the case $c_g < c$, however, individual waves are moving faster than the packet; the lower limit in the X' integral should then be $-\infty$ so that the vorticity is zero downstream of the packet.

To find the phase change in the outer problem, we must obtain the asymptotic behavior of the integrals as $|Y| \to \infty$ in [11.35]. The procedure is given in [MAS 94] with the result being that there is a $-\pi$ phase change (for $\bar{u}_c' > 0$) associated with the logarithm in the ϕ_B series in [11.19]. The asymptotic behavior of the particular integral in [11.35], in fact, includes two terms: one containing $A(X,T)$ and the other A_X. The second of these matches to $\phi_2 A_X$ and it shows that the $\log(y - y_c)$ term in [11.30] has the same phase change as ϕ_1. The analysis of the case $c_g < c$ proceeds in the same way and, despite the lower limit being different for the X' integral, the phase change is again $-\pi\,\mathrm{sgn}(\bar{u}_c')$.

As noted above, [11.34] can also be solved by taking a Fourier transform in X. The difficulty that arises is that there is a pole on the real axis in the inversion integral. To deal with this issue, we look for solutions of the form

$$\hat{\psi} = \phi(X, y, T) e^{ik(x-ct)} e^{\mu\epsilon t}.$$

With $\epsilon > 0$, this has the effect of making the perturbation very small as $t \to -\infty$. The factor of μ is included so that the constant ϵ appears at the order where it is required, i.e. when solving for $\Phi^{(2)}$. Denoting transforms with an overbar, we obtain

$$\bar{\Phi}^{(2)}_{YY} = \frac{i\,k\,\bar{u}_c''\,p}{\bar{u}_c'}\,\frac{\bar{A}}{\lambda - p\,Y - i\,\epsilon/(c - c_g)}\,, \qquad [11.36]$$

where the Fourier transform is defined according to the convention

$$\bar{F}(\lambda, T) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} F(X,T)\,e^{-i\,\lambda X}\,d\,X.$$

The convolution theorem can now be used to invert [11.36] with the inversion contour passing below the singularity at $\lambda = pY$ for $c - c_g > 0$ and above the singularity for $c - c_g < 0$. Once the inversion is accomplished, an expression for $\Phi^{(2)}_{YY}$ is obtained which, in the case $c - c_g > 0$, can be written as

$$\Phi^{(2)}_{YY} = -\frac{k\,\bar{u}_c'' p}{\bar{u}_c'}\int_0^{\infty} A(X - \lambda, T)\,e^{ip\lambda Y}\,d\,\lambda\,. \qquad [11.37]$$

This can be integrated once with respect to Y and expansion of the resulting integral for $|Y| \gg 1$ yields the same phase change as the direct method.

11.4. Nonlinear instabilities governed by integro-differential equations

In this section, a class of instabilities is considered, characterized by an amplification that is quite rapid compared with that predicted by the more familiar weakly nonlinear methods. Typically, the flows are linearly unstable and susceptible to an inviscid instability, so the Reynolds numbers must be large. An excellent example is a pair of slightly supercritical oblique waves on a tanh y mixing layer. In section 11.2.2, the neutral solution for a plane wave with phase speed $c = 0$ is given. It is regular, as is the corresponding oblique wave. However, Goldstein and Choi [GOL 89] pointed out that a perturbation consisting of a pair of oblique waves has a strong critical point singularity with two of the three velocity components behaving as y^{-1} near y_c.

Here we present only the principal features of the analysis in [GOL 89] and note the most significant results. First, the critical layer thickness was shown to be $\varepsilon^{1/3}$, i.e. somewhat larger than $\varepsilon^{1/2}$, the thickness for nonlinear neutral modes. The wave number was taken to be slightly unstable; we will write $k = 1 - \varepsilon^{1/3} k_1$, where k_1 is a constant that is positive and $O(1)$. (The analysis in [GOL 89] was for spatially growing waves, but we will use a temporal viewpoint to be consistent with the rest of this review.) The nonlinear term in the amplitude equation has the same form as [11.1] and [11.15] in that it involves $A|A|^2$, but these terms are now inside a double integral of convolution form. It was shown in [GOL 89] that the amplitude equation develops a singularity in a finite time. The first amplitude equation including such a convolution term was actually derived by Hickernell [HIC 84] for a singular Rossby wave, but he did not solve the equation or recognize that it exhibits an "explosive instability".

The form of the Ginzburg–Landau and the other amplitude equations discussed in sections 11.1–11.3 is determined by a separation of variables in the outer expansion. The latter is an expansion of the velocity perturbation in powers of ε, an amplitude parameter. The critical layer is passive, although its contribution to the coefficients of the amplitude equation can be important. This contrasts greatly with the amplitude equations introduced in this section, where nonlinearity becomes important sooner in the critical layer. Its dynamics then dictate the form of the amplitude equation which is, typically, an integro-differential equation.

11.4.1. *The zonal wave packet critical layer*

To conclude this section, an example from geophysical fluid dynamics will be presented, which combines a number of the ideas described above. We will focus on the nonlinear development of a perturbation to the zonal shear flow $\bar{u} = \tanh y$. The basic flow is to the east (x-direction), y is the north-south coordinate and variations in the vertical are neglected. The vorticity equation, after making the beta-plane approximation, can be written as

$$\zeta_t + \psi_y \, \zeta_x - \psi_x \, \zeta_y - \beta \, \psi_x = R^{-1} \, \nabla^2 \zeta \, . \qquad [11.38]$$

The term multiplied by β is the additional term compared with [11.16]. It represents the Coriolis force, modeled by a linearization about some mean latitude, where β is the derivative of the Coriolis parameter (assumed constant).

Separating variables, as in section 11.3, leads to the Rayleigh–Kuo equation

$$(\bar{u} - c)(\phi'' - k^2 \phi) + (\beta - \bar{u}'')\phi = 0. \qquad [11.39]$$

The linear, neutral solution for the $\tanh y$ mixing layer was obtained in closed form by Howard and Drazin [HOW 64]. The eigenfunction $\phi(y)$ and the conditions relating to the phase speed, wave number and beta parameter are given by

$$\phi = (1 - T)^{\frac{1}{2}(1+c)}(1 + T)^{\frac{1}{2}(1-c)}\,, \quad c^2 = 1 - k^2 \quad \text{and} \quad \beta = -2c(1 - c^2), \qquad [11.40]$$

where $T = \tanh y$. As shown in Figure 11.2, the effect of rotation stabilizes and the range of unstable wave numbers decreases with increasing β until the critical value $\beta = 4/3^{3/2}$ is reached; above this value, the flow is linearly stable.

The existence of a critical value for β makes this flow more suitable for a weakly nonlinear analysis than the $\beta = 0$ mixing layer. Here, it is possible to study the evolution of the fastest growing wave. Churilov and Shukhman [CHU 86] have, in fact, determined the Landau constant a_2 in [11.1] all along the stability boundary. The real part of a_2, which was found to be negative in [SCH 64], changes sign as k^2 decreases and subcritical instability becomes possible for longer waves.

The important role played by viscosity in [CHU 86] calls into question its relevance to planetary atmospheres. As described above, other critical layer balances are possible and one that the author believes is more appropriate to the atmosphere replaces viscosity by wave packet effects. Mallier and Maslowe [MAL 99] reported a weakly nonlinear analysis in which various possibilities were considered. An expansion near the critical value of $\beta = 4/3^{3/2}$ was used in which $\hat{\psi}$ was expanded in powers of both μ and ε. The linear wave packet expansion [11.28] was generalized by writing

$$\hat{\psi} \sim [A(\xi, \tau)\phi(y)\, e^{ik(x-ct)} + *] + \varepsilon\psi^{(1,0)} + \mu\psi^{(0,1)} + \varepsilon\mu\psi^{(1,1)} + \cdots \,, \qquad [11.41]$$

where $\xi = \mu(x - \omega' t)$, $\tau = \mu^2 t$ and the symbol * indicates the complex conjugate. It should be noted that even though $\phi(y)$ in [11.40] is regular, all higher order terms in [11.41] are singular at the critical point.

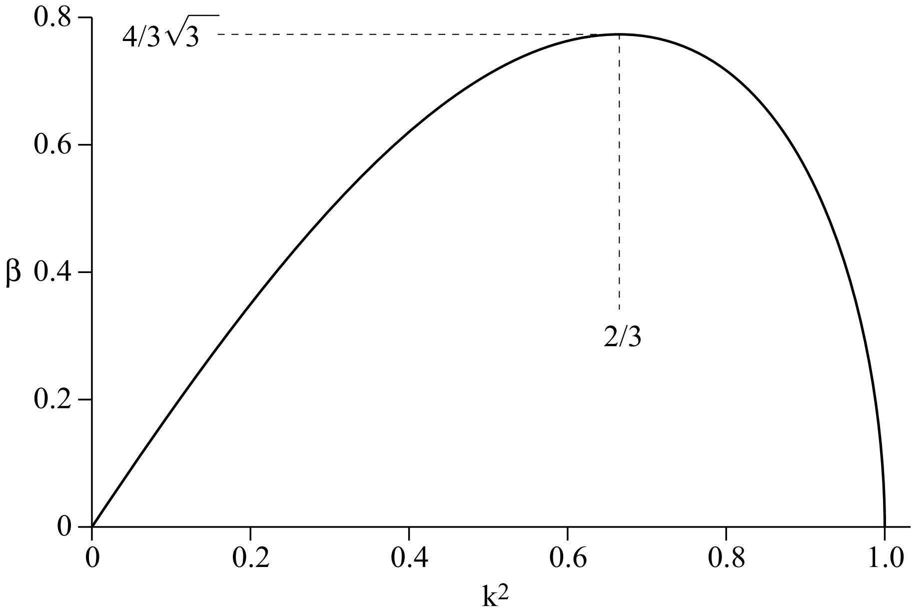

Figure 11.2. *Neutral stability curve for the zonal mixing layer* $\bar{u} = \tanh y$

It was seen in section 11.3 that different balances are possible even on an inviscid basis. The choice $\mu > \varepsilon^{1/2}$ means that a wave packet critical layer, rather than a nonlinear critical layer, is invoked. In that case, the amplitude $A(\xi, \tau)$ was found in [MAL 99] to satisfy the integro-differential equation

$$\sqrt{\frac{2}{3}}\, i\,\gamma_1 \frac{\partial A}{\partial \tau} + \gamma_2 \frac{\partial^2 A}{\partial \xi^2} = \left(\frac{\varepsilon}{\mu^2}\right)^2 \frac{8i}{9} \int_0^\infty \int_0^\infty \xi_0^2 A(\xi + \xi_0, \tau) A(\xi + \xi_0 + \xi_1, \tau) \times A^*_{\xi\xi}(\xi + 2\xi_0 + \xi_1, \tau)\, d\xi_0\, d\xi_1, \qquad [11.42]$$

where the constant γ_1 is real and γ_2 is complex.

The amplitude equation [11.42] can be used to investigate the instability of the mode at the critical β, following the method of section 11.2, by writing

$$A(\xi, \tau) = A_0 + b(\tau) e^{i\kappa\xi}. \qquad [11.43]$$

Again, $b(\tau)$ satisfies a second-order equation and the necessary condition for instability is

$$\kappa^8 > \left(\frac{\varepsilon}{\mu^2}\right)^4 A_0^4 \, \frac{16\,\pi^2}{243\,|\gamma_2|^2} \, . \qquad [11.44]$$

As was the case for Stokes waves, [11.44] shows that modulational instabilities can occur for large enough values of κ, i.e. for packets whose bandwidth is not narrow. However, our example is more like that of Bénard convection in that a linear stability curve exists and we have shown that secondary instability is possible for finite amplitude neutral modes with values of β equal to or less than the critical value.

11.5. Concluding remarks

Linear stability theory has been pursued for more than 100 years and it continues to be an active area of research. Direct numerical simulation is much newer, but its use is already widespread due to the increase in computing power in recent years. Initial conditions and the phenomena investigated by the numerical studies have, in the past, usually been provided by linear theory. The finite amplitude methods discussed here can be much more powerful than linear theory. They suggest new mechanisms of instability, such as resonant interactions, and provide the relevant time scales. In addition, critical layer analyses indicate the resolution requirements for computational schemes. This was discussed in some detail by Maslowe [MAS 86] in 1986. Nine years later, a careful study by Staquet [STA 95] revealed instabilities in the diffusive layers of cats-eye structures in stratified mixing layers. These had been predicted by the critical layer theory, but were not observed in earlier numerical studies because of insufficient resolution.

Large-scale computations can suggest interesting directions for the type of analytical efforts presented in this chapter. Cats-eye patterns are evident in numerical simulations of hurricanes currently being carried out by Professors Yau and Brunet and their students at McGill University. Menelaou *et al.* [MEN 13], for example, have investigated the critical layer interaction of vortices with the quasi-modes known as vortex Rossby waves. The simulations reported in [MEN 13] suggest that this interaction may lead to the intensification of hurricanes. This phenomenon is currently being studied by perturbation methods and the publication of the results is eagerly anticipated.

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