How Does Krein Signature Determine Veering and Crossing of Eigencurves of Non-Conservative Rotating Continua?

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Abstract. Frequency loci crossing and veering phenomena are closely related to wave propagation and instabilities in fluids and structures. In engineering applications the crossings of the eigencurves are typically observed in gyroscopic or potential systems in the presence of symmetries, such as rotational or spherical one. The examples are perfect solids of revolution that serve for modeling turbine wheels, disk and drum brakes, tires, clutches, paper calenders and other rotating machinery. We consider an axi-symmetric flexible rotor perturbed by dissipative, conservative, and non-conservative positional forces originated at the contact with the anisotropic stator. The Campbell diagram of the unperturbed system is a mesh-like structure in the frequency-speed plane with double eigenfrequencies at the nodes. Computing sensitivities of the doublets we find that at every particular node the unfolding of the mesh into the branches of complex eigenvalues in the first approximation is generically determined by only four 2×2 sub-blocks of the perturbing matrix. Selection of the unstable modes that cause self-excited vibrations in the subcritical speed range, is governed by the exceptional points at the corners of the singular eigenvalue surfaces—'double coffee filter' and 'viaduct'—which are sharply associated with the crossings of the unperturbed Campbell diagram with the definite symplectic (Krein) signature. A model of a rotating shaft with two degrees of freedom and a continuous model of a rotating circular string passing through the eyelet are studied in detail.

Keywords: Rotor dynamics, friction, Campbell diagram, eigencurves, veering, stability, Krein signature, brake squeal, exceptional point
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UNFOLDING MACKAY'S CONES UNDER NON-HAMILTONIAN PERTURBATIONS

Bending waves propagate in the circumferential direction of an elastic body of revolution rotating about its axis of symmetry. The frequencies of the waves plotted against the rotational speed are referred to as *the Campbell diagram*. The spectrum of a perfect rotationally symmetric rotor at standstill has infinitely many double semi-simple eigenvalues—*the doublet modes*. By this reason the Campbell diagram contains the eigencurves originated after splitting of the doublets by gyroscopic forces. The branches correspond to simple pure imaginary eigenvalues and intersect each other forming a *spectral mesh* in the frequency-speed plane with the doublets at the nodes. Perturbations of the axially symmetric rotor by dissipative, conservative, and non-conservative positional forces, caused by its contact with the anisotropic stator, generically unfold the spectral mesh of pure imaginary eigenvalues of the Campbell diagram into separate branches of complex eigenvalues in the $(\Omega, \mathrm{Im}\lambda, \mathrm{Re}\lambda)$-space. Nevertheless, the eigencurves in the perturbed Campbell diagram can both avoid crossings and cross each other. Moreover, the real parts of the perturbed eigenvalues plotted against the rotational speed—*decay rate plots*—can also intersect each other and inflate into 'bubbles'. The present work reveals that the unfolding of the Campbell diagrams is determined by a limited number of local scenarios for eigenvalues as a function of parameters, which form stratified manifolds.

Without loss of generality, we consider the finite-dimensional anisotropic rotor system

$$\ddot{\mathbf{x}} + (2\Omega\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + (\mathbf{P} + \Omega^2\mathbf{G}^2 + \kappa\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0, \tag{1}$$

which is a perturbation of the isotropic one

$$\ddot{\mathbf{x}} + 2\Omega\mathbf{G}\dot{\mathbf{x}} + (\mathbf{P} + \Omega^2\mathbf{G}^2)\mathbf{x} = 0, \tag{2}$$

where $\mathbf{x} = \mathbb{R}^{2n}$, $\mathbf{P}$=diag$(\omega_1^2, \omega_1^2, \omega_2^2, \omega_2^2, \dots, \omega_n^2, \omega_n^2)$ is the stiffness matrix, and $\mathbf{G} = -\mathbf{G}^T$ is the gyroscopic matrix

$$\mathbf{G} = \mathrm{blockdiag}(\mathbf{J}, 2\mathbf{J}, \dots, n\mathbf{J}), \quad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

The matrices of non-Hamiltonian perturbation corresponding to velocity-dependent dissipative forces, $\mathbf{D} = \mathbf{D}^T$, and non-conservative positional forces, $\mathbf{N} = -\mathbf{N}^T$, as well as the matrix $\mathbf{K} = \mathbf{K}^T$ of the Hamiltonian perturbation that

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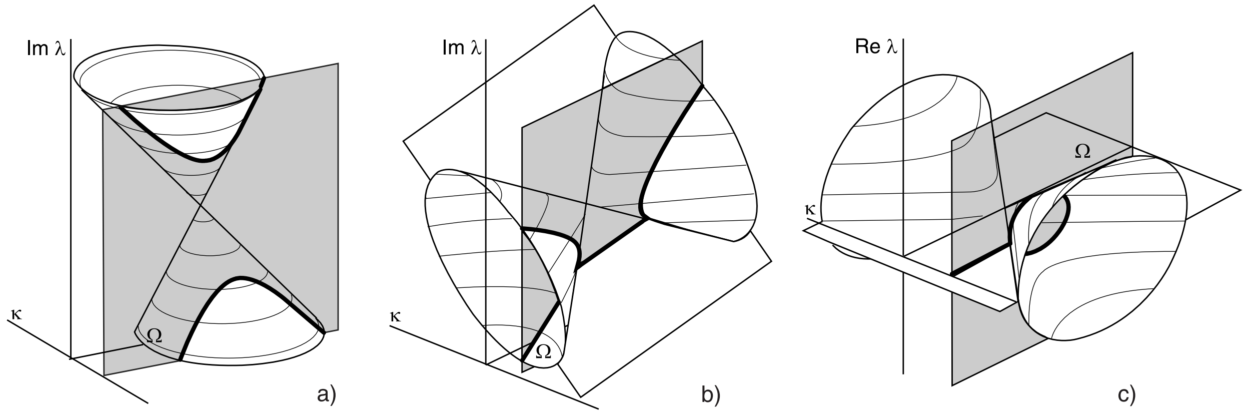

FIGURE 1. MacKay's cones and (bold lines) their cross-sections in the plane $\kappa = const$ (grey): (a) a near-vertically oriented cone $\mathrm{Im}\lambda(\Omega,\kappa)$ for $\alpha\beta > 0$; (b) imaginary parts forming a near-horizontally oriented cone with the attached membrane and (c) the real parts forming a near-horizontally oriented cone $(\mathrm{Re}\lambda)^2 = -\mathrm{Re}c$ with the attached membrane $\mathrm{Re}\lambda = 0$ for $\alpha\beta < 0$.

breaks the rotational symmetry, can depend on the rotational speed Ω. The intensity of the perturbation is controlled by the parameters δ, κ, and ν. Substituting $\mathbf{x} = \mathbf{u}\exp(\lambda t)$ into (2), we arrive at the eigenvalue problem

$$\mathbf{L}_0(\Omega)\mathbf{u} := (\mathbf{I}\lambda^2 + 2\Omega\mathbf{G}\lambda + \mathbf{P} + \Omega^2\mathbf{G}^2)\mathbf{u} = 0. \quad (4)$$

Introducing the indices $\alpha,\beta,\varepsilon,\sigma = \pm 1$ we find that the eigenvalue branches of the operator $\mathbf{L}_0$, $\lambda_s^\varepsilon = i\alpha\omega_s + i\varepsilon s\Omega$ and $\lambda_t^\sigma = i\beta\omega_t + i\sigma t\Omega$, cross each other at $\Omega = \Omega_0$ with the origination of the double semi-simple eigenvalue $\lambda_0 = i\omega_0$

$$\Omega_0 = \frac{\alpha\omega_s - \beta\omega_t}{\sigma t - \varepsilon s}, \quad \omega_0 = \frac{\alpha\sigma\omega_s t - \beta\varepsilon\omega_t s}{\sigma t - \varepsilon s}. \quad (5)$$

Let $\mathbf{M}$ be one of the matrices $\mathbf{D}$, $\mathbf{K}$, or $\mathbf{N}$. Denote $\mathbf{M}_{st} \in \mathbb{R}^{2\times 2}$ a block of the matrix $\mathbf{M} \in \mathbb{R}^{2n\times 2n}$

$$\mathbf{M}_{st} = \begin{pmatrix} m_{2s-1,2t-1} & m_{2s-1,2t} \\ m_{2s,2t-1} & m_{2s,2t} \end{pmatrix}, \quad s,t = 1,2,\ldots,n. \quad (6)$$

We consider perturbation of the matrix operator of the isotropic rotor $\mathbf{L}_0(\Omega) + \Delta\mathbf{L}(\Omega)$. The size of the perturbation $\Delta\mathbf{L}(\Omega) = \delta\lambda\mathbf{D} + \kappa\mathbf{K} + \nu\mathbf{N} \sim \widetilde{\varepsilon}$ is small, where $\widetilde{\varepsilon} = \|\Delta\mathbf{L}(\Omega_0)\|$. Then, the doublet $\lambda_0 = i\omega_0$ splits as

$$\mathrm{Re}\lambda = -\frac{1}{8}\left(\frac{\mathrm{Im}A_1}{\alpha\omega_s} + \frac{\mathrm{Im}B_1}{\beta\omega_t}\right) \pm \sqrt{\frac{|c| - \mathrm{Re}c}{2}}, \quad \mathrm{Im}\lambda = \omega_0 + \frac{\Delta\Omega}{2}(s\varepsilon + t\sigma) + \frac{\kappa}{8}\left(\frac{\mathrm{tr}\mathbf{K}_{ss}}{\alpha\omega_s} + \frac{\mathrm{tr}\mathbf{K}_{tt}}{\beta\omega_t}\right) \pm \sqrt{\frac{|c| + \mathrm{Re}c}{2}}, \quad (7)$$

$$\begin{aligned} \mathrm{Im}c &= \frac{\alpha\omega_t\mathrm{Im}A_1 - \beta\omega_s\mathrm{Im}B_1}{8\omega_s\omega_t}(s\varepsilon - t\sigma)\Delta\Omega + \kappa\frac{(\alpha\omega_s\mathrm{tr}\mathbf{K}_{tt} - \beta\omega_t\mathrm{tr}\mathbf{K}_{ss})(\alpha\omega_s\mathrm{Im}B_1 - \beta\omega_t\mathrm{Im}A_1)}{32\omega_s^2\omega_t^2} \\ &- \alpha\beta\kappa\frac{\mathrm{Re}A_2\mathrm{tr}\mathbf{K}_{st}\mathbf{J}_{\varepsilon\sigma} - \mathrm{Re}B_2\mathrm{tr}\mathbf{K}_{st}\mathbf{I}_{\varepsilon\sigma}}{8\omega_s\omega_t}, \\ \mathrm{Re}c &= \left(\frac{t\sigma - s\varepsilon}{2}\Delta\Omega + \kappa\frac{\beta\omega_s\mathrm{tr}\mathbf{K}_{tt} - \alpha\omega_t\mathrm{tr}\mathbf{K}_{ss}}{8\omega_s\omega_t}\right)^2 - \frac{(\alpha\omega_s\mathrm{Im}B_1 - \beta\omega_t\mathrm{Im}A_1)^2 + 4\alpha\beta\omega_s\omega_t((\mathrm{Re}A_2)^2 + (\mathrm{Re}B_2)^2)}{64\omega_s^2\omega_t^2} \\ &+ \alpha\beta\frac{(\mathrm{tr}\mathbf{K}_{st}\mathbf{J}_{\varepsilon\sigma})^2 + (\mathrm{tr}\mathbf{K}_{st}\mathbf{I}_{\varepsilon\sigma})^2}{16\omega_s\omega_t}\kappa^2. \end{aligned} \quad (8)$$

The coefficients A_1, A_2 and B_1, B_2 depend only on those entries of the matrices $\mathbf{D}$, $\mathbf{K}$, and $\mathbf{N}$ that belong to the four 2×2 blocks (6) with the indices s and t

$$\begin{aligned} A_1 &= \delta\lambda_0\mathrm{tr}\mathbf{D}_{ss} + \kappa\mathrm{tr}\mathbf{K}_{ss} + \varepsilon 2i\nu n_{2s-1,2s}, \quad A_2 = \sigma\nu\mathrm{tr}\mathbf{N}_{st}\mathbf{I}_{\varepsilon\sigma} + i(\delta\lambda_0\mathrm{tr}\mathbf{D}_{st}\mathbf{J}_{\varepsilon\sigma} + \kappa\mathrm{tr}\mathbf{K}_{st}\mathbf{J}_{\varepsilon\sigma}), \\ B_1 &= \delta\lambda_0\mathrm{tr}\mathbf{D}_{tt} + \kappa\mathrm{tr}\mathbf{K}_{tt} + \sigma 2i\nu n_{2t-1,2t}, \quad B_2 = \sigma\nu\mathrm{tr}\mathbf{N}_{st}\mathbf{J}_{\varepsilon\sigma} - i(\delta\lambda_0\mathrm{tr}\mathbf{D}_{st}\mathbf{I}_{\varepsilon\sigma} + \kappa\mathrm{tr}\mathbf{K}_{st}\mathbf{I}_{\varepsilon\sigma}), \end{aligned} \quad (9)$$

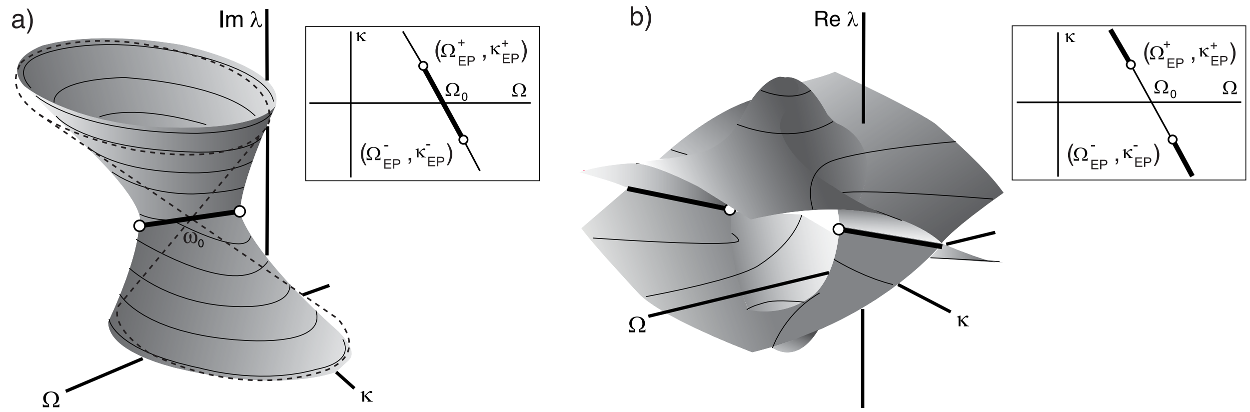

FIGURE 2. (a) The ‘double coffee filter’ singular surface $\mathrm{Im}\lambda(\Omega,\kappa)$ with the exceptional points (open circles) and branch cut (bold lines) originated from the MacKay’s cone (dashed lines) due to mixed dissipative and circulatory perturbation at any crossing with the definite symplectic signature ($\alpha\beta > 0$); (b) the corresponding ‘viaduct’ $\mathrm{Re}\lambda(\Omega,\kappa)$.

where

$$\mathbf{I}_{\varepsilon\sigma} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \sigma \end{pmatrix}, \quad \mathbf{J}_{\varepsilon\sigma} = \begin{pmatrix} 0 & -\sigma \\ \varepsilon & 0 \end{pmatrix}. \tag{10}$$

Assuming $\delta = 0$ and $\nu = 0$ in (7) we find that the eigenvalues of the system (1) with the stiffness modification $\kappa\mathbf{K}$ either are pure imaginary ($\mathrm{Re}\lambda = 0$) and form a conical surface in the $(\Omega,\kappa,\mathrm{Im}\lambda)$-space with the apex at the point $(\Omega_0, 0, \omega_0)$, see Fig. 1(a), or they are complex and in the $(\Omega,\kappa,\mathrm{Re}\lambda)$-space their real parts originate a cone $(\mathrm{Re}\lambda)^2 = -\mathrm{Re}c$ with the apex at the point $(\Omega_0, 0, 0)$, Fig. 1(c). In the $(\Omega,\kappa,\mathrm{Im}\lambda)$-space the corresponding imaginary parts belong to a plane, which is attached to a cone as shown in Fig. 1(b). As it was shown by MacKay (1986), these are the only two possible spatial orientations of the cones corresponding to either definite ($\alpha\beta > 0$) or mixed ($\alpha\beta < 0$) *Krein signature*. The one-parameter slices of the MacKay’s cones give the corresponding Campbell diagram.

In general, dissipative, $\delta\mathbf{D}$, and non-conservative positional, $\nu\mathbf{N}$, perturbations unfold the MacKay’s eigenvalue cones into the surfaces $\mathrm{Im}\lambda(\Omega,\kappa)$ and $\mathrm{Re}\lambda(\Omega,\kappa)$, described by formulas (7). The new eigenvalue surfaces have singularities at the *exceptional points*. The latter correspond to the double eigenvalues with the Jordan chain that born from the parent semi-simple doublet $i\omega_0$ at $\Omega = \Omega_0$, see Fig. 2. The *double coffee filter* and the *viaduct* are the imaginary and the real part of the unfolding of any double pure imaginary semi-simple eigenvalue at the crossing of the Campbell diagram with the definite Krein signature.

Generically, the structure of the perturbing matrices determines only the details of the geometry of the surfaces, such as the coordinates of the exceptional points and the spacial orientation of the branch cuts without qualitative changes. The two eigenvalue surfaces found unite seeming different problems on friction-induced instabilities in rotating elastic continua, because their existence does not depend on the specific model of the rotor-stator interaction. The double coffee filter singularity and its viaduct companion are true symbols of instabilities causing the wine glass to sing and the brake to squeal that connect these phenomena of the wave propagation in rotating continua with the physics of non-Hermitian singularities associated with the wave propagation in stationary anisotropic chiral media.

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