Perturbation of eigenvalues in multiparameter families of non-self-adjoint operator matrices

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Abstract. We consider two-point non-self-adjoint boundary eigenvalue problems for linear matrix differential operators. The coefficient matrices in the differential expressions and the matrix boundary conditions are assumed to depend analytically on the complex spectral parameter λ and on the vector of real physical parameters $\mathbf{p}$. We study perturbations of semi-simple multiple eigenvalues as well as perturbations of non-derogatory eigenvalues under small variations of $\mathbf{p}$. Explicit formulae describing the bifurcation of the eigenvalues are derived. Application to the problem of excitation of unstable modes in rotating elastic continua revealed that selection of the unstable modes in the subcritical speed range, is governed by the exceptional points at the corners of the singular eigenvalue surfaces—'double coffee filter' and 'viaduct'—which are associated with the crossings of the unperturbed Campbell diagram with the definite Krein signature.

1. Introduction

Non-self-adjoint boundary eigenvalue problems for matrix differential operators describe distributed non-conservative systems with the coupled modes and appear in structural mechanics, fluid dynamics, magnetohydrodynamics, to name a few. Practical needs for optimization and rational experiment planning in modern applications allow both the differential expression and the boundary conditions to depend analytically on the spectral parameter and smoothly on several physical parameters (which can be scalar or distributed) [4, 10, 19, 22, 23]. In the multiparameter operator families, eigenvalues with various algebraic and geometric multiplicities can be generic [3]. In some applications additional symmetries yield the existence of *spectral meshes* [23] in the plane 'eigenvalue versus parameter' containing infinite number of nodes with the double eigenvalues (doublets) [8, 11, 18, 19, 21, 23]. As it has been pointed out already by Rellich [2] sensitivity analysis of multiple eigenvalues is complicated by their non-differentiability as functions of several parameters.

An increasing number of multiparameter non-self-adjoint boundary eigenvalue problems and the need for simple constructive estimates of critical parameters and eigenvalues as well as for verification of numerical codes, require development of applicable methods, allowing one to track relatively easily and conveniently the changes in simple and multiple eigenvalues and the corresponding eigenvectors due to variation of the differential expression and especially due to transition from one type of boundary conditions to another one without discretization of the original distributed problem.

A systematical study of bifurcation of eigenvalues of a non-self-adjoint linear operator L_0 due to perturbation $L_0 + \varepsilon L_1$, where ε is a small parameter, dates back to 1950s. Apparently, Krein was the first who derived a formula for the splitting of a double eigenvalue with the Jordan block at the Hamiltonian $1:1$ resonance (*the Krein collision* of the eigenvalues with the opposite *Krein signature* [6]), which was expressed through the generalized eigenvectors of the double eigenvalue [5]. The eigenvalues of the same Krein signature avoid collisions under Hamiltonian perturbations. Recent works on friction-induced oscillations in rotating continua [8, 11, 18, 19, 21, 23] have rosen a question on the instabilities caused by the interacting eigenvalues of the same Krein type in non-self-adjoint boundary eigenvalue problems for operator matrices when the perturbation is non-Hamiltonian.

2. Mathematical setting

We consider the boundary eigenvalue problem

$$\mathbf{L}(\lambda, \mathbf{p})\mathbf{u} = 0, \quad \mathbf{U}_k(\lambda, \mathbf{p})\mathbf{u} = 0, \quad k = 1, \ldots, m, \tag{1}$$

where $\mathbf{u}(x) \in \mathbb{C}^N \otimes C^{(m)}[0,1]$. The differential expression $\mathbf{Lu}$ of the operator is

$$\mathbf{Lu} = \sum_{j=0}^{m} \mathbf{l}_j(x)\partial_x^{m-j}\mathbf{u}, \quad \mathbf{l}_j(x) \in \mathbb{C}^{N\times N} \otimes C^{(m-j)}[0,1], \quad \det[\mathbf{l}_0(x)] \neq 0, \tag{2}$$

and the boundary forms $\mathbf{U}_k\mathbf{u}$ are

$$\mathbf{U}_k\mathbf{u} = \sum_{j=0}^{m-1} \mathbf{A}_{kj}\mathbf{u}_x^{(j)}(x=0) + \sum_{j=0}^{m-1} \mathbf{B}_{kj}\mathbf{u}_x^{(j)}(x=1), \quad \mathbf{A}_{kj}, \mathbf{B}_{kj} \in \mathbb{C}^{N\times N}. \tag{3}$$

Introducing the block matrix $\mathfrak{U} := [\mathfrak{A}, \mathfrak{B}] \in \mathbb{C}^{mN\times 2mN}$ and the vector

$$\mathfrak{u}^T := \left(\mathbf{u}^T(0), \mathbf{u}_x^{(1)T}(0), \ldots, \mathbf{u}_x^{(m-1)T}(0), \mathbf{u}^T(1), \mathbf{u}_x^{(1)T}(1), \ldots, \mathbf{u}_x^{(m-1)T}(1)\right) \in \mathbb{C}^{2mN} \tag{4}$$

the boundary conditions can be rewritten as $\mathfrak{U}\mathfrak{u} = [\mathfrak{A}, \mathfrak{B}]\mathfrak{u} = 0$, where $\mathfrak{A} = (\mathbf{A}_{kj})|_{x=0} \in \mathbb{C}^{mN\times mN}$ and $\mathfrak{B} = (\mathbf{B}_{kj})|_{x=1} \in \mathbb{C}^{mN\times mN}$. It is assumed that the matrices $\mathbf{l}_j$, $\mathfrak{A}$, and $\mathfrak{B}$ are analytic functions of the complex spectral parameter λ and smooth functions of the real vector of physical parameters $\mathbf{p} \in \mathbb{R}^n$.

Let us introduce a scalar product $< \mathbf{u}, \mathbf{v} > := \int_0^1 \mathbf{v}^*\mathbf{u}dx$, where the asterisk denotes complex-conjugate transpose ($\mathbf{v}^* := \overline{\mathbf{v}}^T$). Taking the scalar product of $\mathbf{Lu}$ and a vector-function $\mathbf{v}$ and integrating it by parts yields the Lagrange formula for the case of operator matrices (cf. [16])

$$\Omega(\mathbf{u}, \mathbf{v}) := < \mathbf{Lu}, \mathbf{v} > - < \mathbf{u}, \mathbf{L}^\dagger\mathbf{v} > = \mathfrak{v}^*\mathcal{L}\mathfrak{u}, \tag{5}$$

with the adjoint differential expression $\mathbf{L}^\dagger\mathbf{v} := \sum_{q=0}^{m}(-1)^{m-q}\partial_x^{m-q}\left(\mathbf{l}_q^*\mathbf{v}\right)$, the vector $\mathfrak{v}$

$$\mathfrak{v}^T := \left(\mathbf{v}^T(0), \mathbf{v}_x^{(1)T}(0), \ldots, \mathbf{v}_x^{(m-1)T}(0), \mathbf{v}^T(1), \mathbf{v}_x^{(1)T}(1), \ldots, \mathbf{v}_x^{(m-1)T}(1)\right) \in \mathbb{C}^{2mN} \tag{6}$$

and the block matrix $\mathcal{L} := (\mathbf{l}_{ij})$

$$\mathcal{L} = \begin{pmatrix} -\mathfrak{L}(0) & 0 \\ 0 & \mathfrak{L}(1) \end{pmatrix}, \quad \mathfrak{L}(x) = \begin{pmatrix} \mathbf{l}_{00} & \mathbf{l}_{01} & \cdots & \mathbf{l}_{0m-2} & \mathbf{l}_{0m-1} \\ \mathbf{l}_{10} & \mathbf{l}_{11} & \cdots & \mathbf{l}_{1m-2} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \mathbf{l}_{m-20} & \mathbf{l}_{m-21} & \cdots & 0 & 0 \\ \mathbf{l}_{m-10} & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{7}$$

where the matrices $\mathbf{l}_{ij}$ are

$$\mathbf{l}_{ij} \;:=\; \sum_{k=i}^{m-1-j} (-1)^k M_{ij}^k\, \partial_x^{k-i} \mathbf{l}_{m-1-j-k},$$
$$M_{ij}^k \;:=\; \begin{cases} \frac{k!}{(k-i)!i!}, & i+j \le m-1 \quad \cap \quad k \ge i \ge 0 \\ 0, & i+j > m-1 \quad \cup \quad k < i. \end{cases} \tag{8}$$

Extend the original matrix $\mathfrak{U}$ to a square matrix $\mathcal{U}$, which is made non-degenerate in a neighborhood of the point $\mathbf{p} = \mathbf{p}_0$ and the eigenvalue $\lambda = \lambda_0$, to which the eigenvector $\mathbf{u}_0$ corresponds, by an appropriate choice of the auxiliary matrices $\widetilde{\mathfrak{A}}(\lambda, \mathbf{p})$ and $\widetilde{\mathfrak{B}}(\lambda, \mathbf{p})$

$$\mathfrak{U} = [\mathfrak{A}, \mathfrak{B}] \hookrightarrow \mathcal{U} := \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \widetilde{\mathfrak{A}} & \widetilde{\mathfrak{B}} \end{pmatrix} \in \mathbb{C}^{2mN\times 2mN}, \quad \widetilde{\mathfrak{U}} := [\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}], \quad \det(\mathcal{U}) \neq 0. \tag{9}$$

For the adjoint boundary conditions $\mathfrak{V}\mathfrak{v} = [\mathfrak{C}, \mathfrak{D}]\mathfrak{v} = 0$ the similar process yields

$$\mathfrak{V} := [\mathfrak{C}, \mathfrak{D}] \hookrightarrow \mathcal{V} := \begin{pmatrix} \mathfrak{C} & \mathfrak{D} \\ \widetilde{\mathfrak{C}} & \widetilde{\mathfrak{D}} \end{pmatrix} \in \mathbb{C}^{2mN\times 2mN}, \quad \widetilde{\mathfrak{V}} := [\widetilde{\mathfrak{C}}, \widetilde{\mathfrak{D}}], \quad \det(\mathcal{V}) \neq 0. \tag{10}$$

Then, the form in (5) can be represented as $\Omega(\mathbf{u}, \mathbf{v}) = (\mathfrak{V}\mathfrak{v})^*\widetilde{\mathfrak{U}}\mathfrak{u} - (\widetilde{\mathfrak{V}}\mathfrak{v})^*\mathfrak{U}\mathfrak{u}$.

3. Perturbation of eigenvalues

Assume that in the neighborhood of the point $\mathbf{p} = \mathbf{p}_0$ the spectrum of the boundary eigenvalue problem (1) is discrete. Denote $\mathbf{L}_0$=$\mathbf{L}(\lambda_0, \mathbf{p}_0)$ and $\mathfrak{U}_0$=$\mathfrak{U}(\lambda_0, \mathbf{p}_0)$. Consider a smooth perturbation of parameters $\mathbf{p} = \mathbf{p}(\varepsilon)$ where $\mathbf{p}(0) = \mathbf{p}_0$ and ε is a small real number. Then, as in the case of analytic matrix functions [12, 15], the Taylor decomposition of the differential operator matrix $\mathbf{L}(\lambda, \mathbf{p}(\varepsilon))$ and the matrix of the boundary conditions $\mathfrak{U}(\lambda, \mathbf{p}(\varepsilon))$ are [14, 16]

$$\mathbf{L}\left(\lambda, \mathbf{p}(\varepsilon)\right) = \sum_{r,s=0}^{\infty} \frac{(\lambda-\lambda_0)^r}{r!}\varepsilon^s\, \mathbf{L}_{rs}, \quad \mathfrak{U}(\lambda, \varepsilon) = \sum_{r,s=0}^{\infty} \frac{(\lambda-\lambda_0)^r}{r!}\varepsilon^s \mathfrak{U}_{rs}, \tag{11}$$

with $\mathbf{L}_{00} = \mathbf{L}_0$, $\mathfrak{U}_{00} = \mathfrak{U}_0$, and

$$\mathbf{L}_{r0} = \partial_\lambda^r \mathbf{L}, \quad \mathfrak{U}_{r0} = \partial_\lambda^r \mathfrak{U}; \quad \mathbf{L}_{r1} = \sum_{j=1}^{n} \dot{p}_j\, \partial_\lambda^r \partial_{p_j} \mathbf{L}, \quad \mathfrak{U}_{r1} = \sum_{j=1}^{n} \dot{p}_j\, \partial_\lambda^r \partial_{p_j} \mathfrak{U},$$

where dot denotes differentiation with respect to ε at $\varepsilon = 0$ and partial derivatives are evaluated at $\mathbf{p}$=$\mathbf{p}_0$, λ=λ_0.

Let at the point $\mathbf{p} = \mathbf{p}_0$ the spectrum contain a semi-simple μ-fold eigenvalue λ_0 with μ linearly-independent eigenvectors $\mathbf{u}_0(x)$, $\mathbf{u}_1(x)$, $\ldots$, $\mathbf{u}_{\mu-1}(x)$. Then, the perturbed eigenvalue $\lambda(\varepsilon)$ and the eigenvector $\mathbf{u}(\varepsilon)$ are [7, 12, 14]

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots, \quad \mathbf{u} = \mathbf{b}_0 + \varepsilon\mathbf{b}_1 + \varepsilon^2\mathbf{b}_2 + \ldots. \tag{12}$$

Substituting expansions (11) and (12) into (1) we find that the coefficients λ_1 are generically μ distinct roots of the μ-th order polynomial

$$\det(\mathbf{F} + \lambda_1\mathbf{G}) = 0. \tag{13}$$

The entries of the $\mu \times \mu$ matrices $\mathbf{F}$ and $\mathbf{G}$ are defined by the expressions

$$F_{ij} =< \mathbf{L}_{01}\mathbf{u}_j, \mathbf{v}_i > + \, \mathfrak{v}_i^* \widetilde{\mathfrak{V}}_0^* \mathfrak{U}_{01} \mathfrak{u}_j, \quad G_{ij} =< \mathbf{L}_{10}\mathbf{u}_j, \mathbf{v}_i > + \, \mathfrak{v}_i^* \widetilde{\mathfrak{V}}_0^* \mathfrak{U}_{10} \, \mathfrak{u}_j. \tag{14}$$

For $\mu = 1$ the formulas (13) and (14) describe a simple eigenvalue

$$\lambda = \lambda_0 - \varepsilon \frac{< \mathbf{L}_{01}\mathbf{u}_0, \mathbf{v}_0 > + \, \mathfrak{v}_0^* \widetilde{\mathfrak{V}}_0^* \mathfrak{U}_{01} \mathfrak{u}_0}{< \mathbf{L}_{10}\mathbf{u}_0, \mathbf{v}_0 > + \, \mathfrak{v}_0^* \widetilde{\mathfrak{V}}_0^* \mathfrak{U}_{10} \, \mathfrak{u}_0} + o(\varepsilon). \tag{15}$$

The formulas (13)–(15) extend the results of [7, 12, 14] to the case of the multiparameter non-self-adjoint boundary eigenvalue problems for operator matrices.

Let at the point $\mathbf{p} = \mathbf{p}_0$ the spectrum contain a non-derogatory μ-fold eigenvalue λ_0 with the Keldysh chain of length μ, consisting of the eigenvector $\mathbf{u}_0(x)$ and the associated vectors $\mathbf{u}_1(x), \ldots, \mathbf{u}_{\mu-1}(x)$ that solve the boundary value problems

$$\mathbf{L}_0\mathbf{u}_0 = 0, \quad \mathfrak{U}_0 \mathfrak{u}_0 = 0, \tag{16}$$

$$\mathbf{L}_0\mathbf{u}_j = -\sum_{r=1}^{j} \frac{1}{r!} \partial_\lambda^r \mathbf{L}\mathbf{u}_{j-r}, \quad \mathfrak{U}_0 \mathfrak{u}_j = -\sum_{r=1}^{j} \frac{1}{r!} \partial_\lambda^r \mathfrak{U} \mathfrak{u}_{j-r}. \tag{17}$$

Substituting into equations (1) the Newton-Puiseux series for the perturbed eigenvalue $\lambda(\varepsilon)$ and eigenvector $\mathbf{u}(\varepsilon)$ [16]

$$\lambda = \lambda_0 + \lambda_1 \varepsilon^{1/\mu} + \ldots, \quad \mathbf{u} = \mathbf{u}_0 + \mathbf{w}_1 \varepsilon^{1/\mu} + \ldots, \tag{18}$$

and collecting terms with the same powers of ε, yields the coefficient λ_1 in (18). Hence, generalizing the results of the works [14, 16], we find

$$\lambda = \lambda_0 + \sqrt[\mu]{-\varepsilon \frac{< \mathbf{L}_{01}\mathbf{u}_0, \mathbf{v}_0 > + \, \mathfrak{v}_0^* \widetilde{\mathfrak{V}}_0^* \mathfrak{U}_{01} \mathfrak{u}_0}{\sum_{r=1}^{\mu} \frac{1}{r!} (< \mathbf{L}_{r0}\mathbf{u}_{\mu-r}, \mathbf{v}_0 > + \, \mathfrak{v}_0^* \widetilde{\mathfrak{V}}_0^* \mathfrak{U}_{r0} \mathfrak{u}_{\mu-r})}} + o(\varepsilon^{\frac{1}{\mu}}), \tag{19}$$

where $\mathbf{v}_0$ solves the adjoint boundary value problem $\mathbf{L}_0^\dagger \mathbf{v}_0 = 0, \quad \mathfrak{V}_0 \mathfrak{v}_0 = 0$.

4. Unfolding the doublets of definite Krein type in the Campbell diagrams

Consider a circular string of displacement $W(\psi, \tau)$, radius r, and mass per unit length ρ that rotates with the speed γ and passes at $\psi = 0$ through a massless eyelet supported by the spring with the stiffness K and damper with the damping coefficient D, Fig. 1(a). With the non-dimensional parameters

$$t = \frac{\tau}{r}\sqrt{\frac{P}{\rho}}, \quad w = \frac{W}{r}, \quad \Omega = \gamma r \sqrt{\frac{\rho}{P}}, \quad k = \frac{Kr}{P}, \quad d = \frac{D}{\sqrt{\rho P}}, \quad \varphi = \frac{\psi}{2\pi}, \tag{20}$$

and assuming $w(\varphi, t) = u(\varphi)\exp(\lambda t)$ we arrive at the non-self-adjoint boundary eigenvalue problem for a matrix ($N = 2, m = 1$) differential operator [9]

$$\mathbf{L}\mathbf{u} := \mathbf{l}_0 \partial_\varphi \mathbf{u} + \mathbf{l}_1 \mathbf{u} = 0, \quad \mathfrak{U}\mathfrak{u} := [\mathfrak{A}, \mathfrak{B}]\mathfrak{u} = 0, \tag{21}$$

where

$$\mathbf{l}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1-\Omega^2 \end{pmatrix}, \quad \mathbf{l}_1 = -\begin{pmatrix} 0 & 1 \\ 4\pi^2\lambda^2 & 4\pi\Omega\lambda \end{pmatrix},$$

$$\mathfrak{A} = \begin{pmatrix} 1 & 0 \\ \frac{2\pi(k+\lambda d)}{\Omega^2-1} & 1 \end{pmatrix}, \quad \mathfrak{B} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

Parameters Ω, d, and k express the speed of rotation, damping, and stiffness.

For $k = 0$ and $d = 0$ the eigenvalues $\lambda_n^{\pm} = in(1 \pm \Omega)$, $n \in \mathbb{Z}$, form the spectral mesh of the Campbell diagram [1, 19] in the plane $(\Omega, \mathrm{Im}\lambda)$, Fig. 1(b). The lines $\lambda_n^{\varepsilon} = in(1 + \varepsilon\Omega)$ and $\lambda_m^{\delta} = im(1 + \delta\Omega)$, where $\varepsilon, \delta = \pm 1$, intersect each other at the node (Ω_0, ω_0) with

$$\Omega_0 = \frac{n - m}{m\delta - n\varepsilon}, \quad \omega_0 = \frac{nm(\delta - \varepsilon)}{m\delta - n\varepsilon}, \tag{23}$$

where the double eigenvalue $\lambda_0 = i\omega_0$ has two orthogonal eigenvectors

$$\mathbf{u}_n^{\varepsilon} = \begin{pmatrix} 1 \\ -i\varepsilon 2\pi n \end{pmatrix} e^{-i\varepsilon 2\pi n\varphi}, \quad \mathbf{u}_m^{\delta} = \begin{pmatrix} 1 \\ -i\delta 2\pi m \end{pmatrix} e^{-i\delta 2\pi m\varphi}. \tag{24}$$

Taking into account that $\delta = -\varepsilon$ at all the crossings, excluding $(\Omega_0 = \pm 1, \omega_0 = 0)$ where $\delta = \varepsilon$, from (13) and (14) we find the real and imaginary parts of the perturbed non-zero double eigenvalues

$$\mathrm{Re}\lambda = -d\frac{n+m}{8\pi nm}\omega_0 \pm \sqrt{\frac{|c| - \mathrm{Re}c}{2}}, \quad \mathrm{Im}\lambda = \omega_0 + \varepsilon\frac{n-m}{2}\Delta\Omega + \frac{n+m}{8\pi nm}k \pm \sqrt{\frac{|c| + \mathrm{Re}c}{2}}, \tag{25}$$

where $\Delta\Omega = \Omega - \Omega_0$, and for the complex coefficient c we have

$$\begin{aligned} \mathrm{Im}c &= k\frac{2d\omega_0}{16\pi^2 nm} + d\omega_0\frac{m-n}{4\pi nm}\left(\varepsilon\frac{n+m}{2}\Delta\Omega + \frac{m-n}{8\pi nm}k\right), \\ \mathrm{Re}c &= \left(\frac{\varepsilon n - \delta m}{2}\Delta\Omega + \frac{m-n}{8\pi nm}k\right)^2 + \frac{k^2}{16\pi^2 nm} - \frac{[d(m+n)\omega_0]^2}{64\pi^2 n^2 m^2}. \end{aligned} \tag{26}$$

Setting $\mathrm{Re}c = 0$ and $\mathrm{Im}c = 0$ we find the coordinates of the projections of the exceptional points (where the perturbed eigenvalue is double and non-derogatory) of the surfaces $\mathrm{Re}\lambda(\Omega, k)$ and $\mathrm{Im}\lambda(\Omega, k)$ onto the (Ω, k)-plane

$$\Omega_{EP} = \Omega_0 \pm \frac{\varepsilon}{8\pi nm}\frac{d(m+n)\omega_0}{\sqrt{nm}}, \quad \kappa_{EP} = \pm\frac{d(n-m)\omega_0}{2\sqrt{nm}}. \tag{27}$$

The existence of the exceptional points (27) depends on the Krein signature [5, 6, 23] of the intersecting branches of the unperturbed Campbell diagram, that is on the sign of the product nm, where $n, m \in \mathbb{Z} - \{0\}$. In the case of the rotating string all the crossings in the subcritical speed range ($|\Omega| < 1$) have definite Krein signature ($nm > 0$). For those in the supercritical speed range ($|\Omega| > 1$) it is mixed with $nm < 0$. In the (Ω, κ)-plane the exceptional points are situated on the line $\mathrm{Im}c = 0$.

Approximations (25) to the eigenvalue surfaces of a string with $d = 0.3$ are presented in Fig. 1. The surface of the imaginary parts shown in Fig. 1(c) is formed by the two Whitney's umbrellas [3, 15, 20] with the handles (branch cuts) glued when they are oriented toward each other. This singular surface is known in the physical literature on wave propagation in anisotropic media as the *double coffee filter* [13, 17]. The *viaduct* singular surface of the real parts results from the gluing of the roofs of two Whitney's umbrellas when their handles are oriented outwards, Fig. 1(d). The double coffee filter singularity is a result of the deformation of the MacKay's eigenvalue cone [6] by the dissipative perturbation. This perturbation foliates the plane $\mathrm{Re}\lambda = 0$ into the viaduct singular surface which has self-intersections along the two branch cuts and an ellipse-shaped arch between the two exceptional points, Fig. 1(d). Both types of singular surfaces appear when non-Hermitian perturbation of Hermitian matrices is considered [17]. The smaller inclusions in Fig. 1 show the cross-sections of the surfaces by the plane $k = 0$ for the convenience of comparing with the corresponding numerical data of [9]. The results shown in Fig. 1 perfectly agree with the numerical modeling of [9].

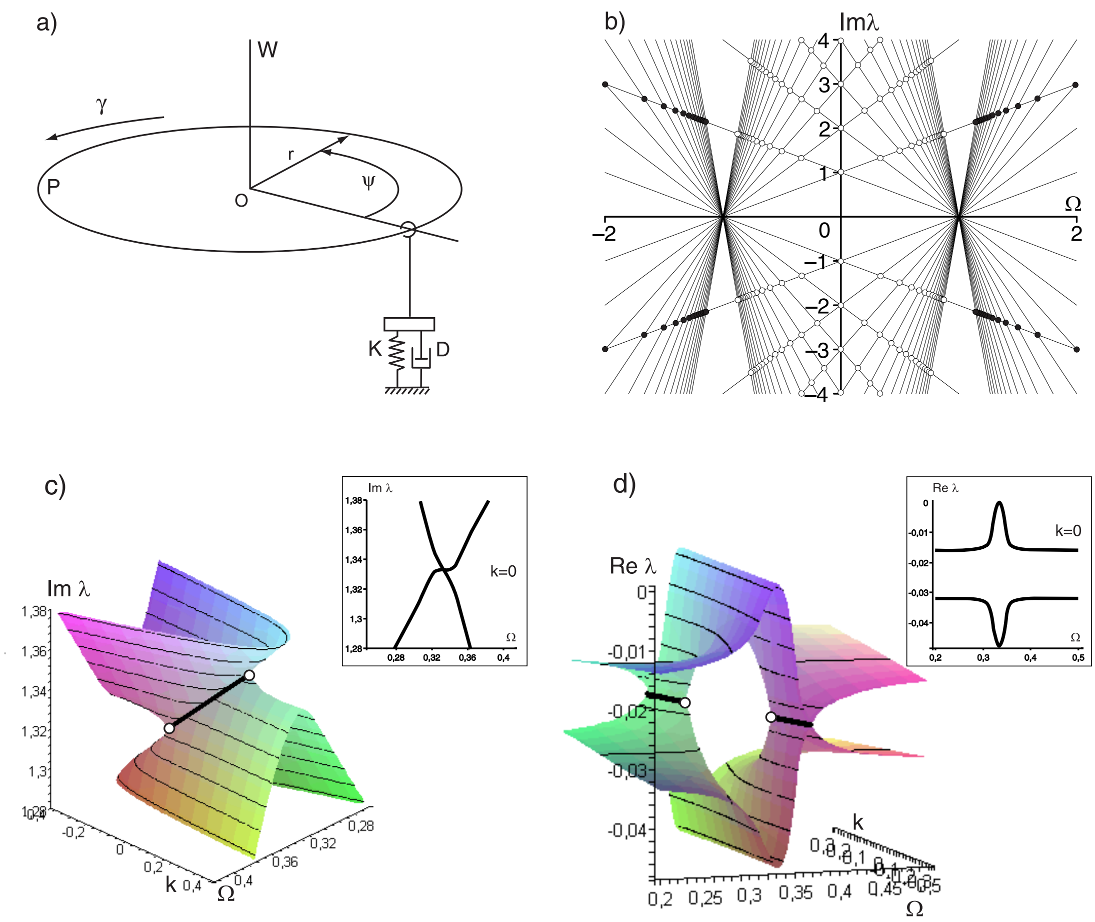

Figure 1. (a) A rotating circular string; (b) 30 modes of its spectral mesh; (c) the *double coffee filter* singular surface $\mathrm{Im}\lambda(\Omega, k)$ in the vicinity of the crossing with $n = 1$ and $m = 2$; (d) the *viaduct* singular surface $\mathrm{Re}\lambda(\Omega, k)$ corresponding to the crossing with $n = 1$ and $m = 2$.

5. Conclusion

A perturbative approach to multiparameter non-self-adjoint boundary eigenvalue problems for operator matrices is developed in the form convenient for implementation in the computer algebra systems for an automatic calculation of the adjoint boundary conditions and coefficients in the perturbation series for simple and multiple eigenvalues and their eigenvectors. The approach is aimed at the applications requiring frequent switches from one set of boundary conditions to another. Applying it to the problem of the onset of instability in rotating continua under symmetry-breaking perturbations, we found that in a weakly anisotropic rotor system the branches of the Campbell diagram and the decay rate plots in the subcritical speed range are the cross-sections of the two companion singular eigenvalue surfaces. The double coffee filter and the viaduct are the imaginary and the real part of the unfolding of any double pure imaginary semi-simple eigenvalue at the crossing of the Campbell diagram with the definite Krein signature. The double coffee filter singularity and its viaduct companion are true symbols of instabilities causing the wine glass to sing and the brake to squeal that connect these phenomena of the wave propagation in rotating continua with the physics of non-Hermitian singularities associated with the wave propagation in stationary anisotropic chiral media.

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