How do small velocity-dependent forces (de)stabilize a non-conservative system?

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Abstract

The influence of small velocity-dependent forces on the stability of a linear autonomous non-conservative system of general type is studied. The problem is investigated by two approaches based on the sensitivity analysis of multiple eigenvalues. In the first case the eigenvalues of a quadratic matrix pencil as functions of a vector of the system parameters are studied. The second approach deals with the roots of the characteristic polynomial whose coefficients are expressed by means of the invariants of matrices of a non-conservative system.

An explicit asymptotic expression for the critical flutter load as a function of parameters corresponding to the velocity-dependent forces is derived. Approximations of the domain of asymptotic stability in the space of parameters are obtained. The behaviour of eigenvalues in the complex plane is investigated and interpreted. The classical problems by H. Ziegler, V.V. Bolotin and G. Herrmann considered as mechanical applications demonstrate the applicability of the developed methods based on the analysis of multiple eigenvalues.

1 Introduction

The dependence of stability of a linear autonomous mechanical system on the structure of forces acting on the system is a classical subject that goes back to the works

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by W. Thomson and P.G. Tait [1]. However, the general interest to the problem of the influence of small velocity-dependent forces on the stability of a linear autonomous non-conservative system arose in the early 1950-s due to the work by H. Ziegler [2]. The first systematic results on non-conservative stability problems in mechanics were summarized in the book by V.V. Bolotin [3] and in the paper by G. Herrmann [4]. Other references on this subject one can find in the detailed surveys by A.P. Seyranian [5] and A.M. Bloch et al. [6]. A recent paper by V.V. Bolotin et al. [7] contains a short review of achievements in the field.

We consider oscillations of a linear autonomous non-conservative mechanical system with m degrees of freedom described by the equation

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2}+\mathbf{D}\frac{d\mathbf{y}}{dt}+\mathbf{A}\mathbf{y}=0,$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{A}$ are real $m\times m$ matrices of inertia, damping and gyroscopic forces as well as non-conservative positional forces, respectively, and $\mathbf{y}$ is an m–dimensional vector of generalized coordinates.

In the absence of the velocity-dependent forces described by the matrix $\mathbf{D}$ the non-conservative system can never be asymptotically stable, but it can be marginally stable oscillating with the limited amplitude. If the system depends on parameters one can subdivide the parameter space into the domains where the system is stable or unstable. How are the domain of marginal stability and the domain of asymptotic stability after the introduction of velocity-dependent forces related to each other?

H. Ziegler [2] discovered the destabilizing influence of such forces on the non-conservative system: The critical load of the Ziegler pendulum decreases in a discontinuous manner with the introduction of a small damping. This effect known as the destabilization paradox was discussed in a number of papers [2–11].

However, already in 1960-s the existence of stabilizing damping configurations for linear non-conservative systems was shown for some specific cases [3, 12, 13]. In the works [10, 11, 14] the first attempts to generalize these results studying bifurcations of eigenvalues of the system were made. Good perspectives of this approach and the necessity of its further development were noted independently in [15, 16]. A multiparameter sensitivity analysis of multiple eigenvalues created in [17, 18] allowed to explain the destabilization paradox and find the domain of stabilization in the plane of damping parameters for the Ziegler pendulum [19, 20]. In the works [21, 22] the domain of stabilization was partially found for general 2×2 non-conservative systems by means of the sensitivity analysis of simple roots of the characteristic polynomial. Recently it was established that the destabilization paradox is close connected with the singularities of the asymptotic stability domain [23], which theory goes back to the works by V.I. Arnold [24].

The goal of the present paper is to obtain the applicable results on the (de)stabilization of non-conservative systems in the general case. The paper is organized in the following way: In Section 2 we study a non-conservative system with m degrees of freedom, assuming that the matrix of damping and gyroscopic forces smoothly depends on a vector of parameters $\mathbf{k}$ and the matrix of non-conservative positional forces is a smooth function of a vector $\mathbf{q}$. We derive the explicit formulae approximating the domain of asymptotic stability in the space of the parameters $\mathbf{k}$ and $\mathbf{q}$. In Section 3 the Herrmann-Jong pendulum is considered in detail. In Section 4 for systems with two degrees of freedom we construct approximations of the domain of asymptotic stability using matrix invariants and find the structure of the matrix of velocity-dependent forces stabilizing a circulatory system. As mechanical examples the Bolotin problem and the problem of Herrman and Jong are considered in detail.

Most of the mathematical results needed for the analysis of stability are concentrated in three appendices. In Appendix A we obtain explicit formulae describing splitting of a multiple eigenvalue of a linear operator whose coefficients smoothly depend on the spectral parameter and a vector of real parameters. In Appendix B we apply the Leverrier-Faddejev algorithm to a 2×2 block matrix in order to express the characteristic polynomial of a quadratic matrix pencil through the invariants of its matrices. In Appendix C we are interested in stable complex polynomials. All the results from the Appendices A, B, C are used throughout the paper.

2 Analysis of eigenvalue problem

Consider a linear autonomous non-conservative mechanical system for free vibrations

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{D}(\mathbf{k})\frac{d\mathbf{y}}{dt} + \mathbf{A}(\mathbf{q})\mathbf{y} = 0, \tag{2.1}$$

where the real $m \times m$ matrices $\mathbf{D}$ and $\mathbf{A}$ are smooth functions of the vectors of parameters $\mathbf{k} \in R^{\mu}$ and $\mathbf{q} \in R^{n-\mu}$, respectively, and besides $\mathbf{D}(0) = 0$. The real $m \times m$ matrix $\mathbf{M}$ is assumed to be constant. The integer index μ takes its values from the interval $0 \le \mu \le n$. Thus, the vector of parameters consists of two independent components $\mathbf{p} = (\mathbf{k}, \mathbf{q}) \in R^n$. Finding the solution of Eq.(2.1) in the form $\mathbf{y}$=$\mathbf{u}\exp(\lambda t)$, where t is time, we get the eigenvalue problem

$$\mathbf{L}\mathbf{u} = 0, \quad \mathbf{L} = \lambda^2\mathbf{M} + \lambda\mathbf{D}(\mathbf{k}) + \mathbf{A}(\mathbf{q}), \tag{2.2}$$

where $\mathbf{u}$ is an eigenvector and λ is an eigenvalue of the linear operator $\mathbf{L}$ for the fixed vectors $\mathbf{k}$ and $\mathbf{q}$.

The non-conservative system without gyroscopic and damping forces ($\mathbf{D} = 0$)

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{A}(\mathbf{q})\mathbf{y} = 0 \tag{2.3}$$

is usually called *circulatory* [2, 3]. Let the bar over a symbol indicates complex conjugation. The spectrum of a circulatory system has a reversible symmetry: If λ is an eigenvalue of (2.2), then $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$ are the eigenvalues too. As a consequence, a circulatory system is stable in the sense of Lyapunov only if all the eigenvalues λ of eigenvalue problem (2.2) are purely imaginary and simple or semi-simple. If with a change of the vector of parameters $\mathbf{q}$ at least one eigenvalue becomes positive and real then the circulatory system loses stability statically (divergence). In the case when there exists a complex-conjugate pair of eigenvalues a dynamic instability (flutter) occurs [2, 3, 25].

It is known that in the case of the general position the smooth parts of the boundary of the stability domain of a circulatory system are made up of surfaces of codimension 1, at whose points the operator $\mathbf{L}$ contains either a double zero eigenvalue or a double purely imaginary eigenvalue with a Jordan chain of length 2, all other eigenvalues λ being purely imaginary and simple. Generally speaking, the stability boundary is not smooth and may have singularities of higher codimension, corresponding to eigenvalues with more complicated Jordan structure [24, 25, 26].

Perturbation of a circulatory system by small velocity-dependent forces destroys the reversible symmetry of eigenvalues. As a result, the non-conservative system can become unstable or asymptotically stable, if all the eigenvalues move into the left-hand half of the complex plane. How should one change the parameters $\mathbf{k}$, $\mathbf{q}$ to make non-conservative system (2.1) asymptotically stable or to destabilize it?

Bifurcation of a double eigenvalue in the generic case. The main tool in our investigation of stability of system (2.1) will be the theory of bifurcation of multiple eigenvalues of linear non-selfadjoint operators going back to the work [27]. We will use here the explicit formulae describing splitting of multiple eigenvalues of a linear matrix operator $\mathbf{L}(\lambda, \mathbf{p})$ smoothly dependent on the complex spectral parameter λ and the vector of real parameters $\mathbf{p} \in R^n$ derived in Appendix A.

Consider a point $\mathbf{p}_0 = (0, \mathbf{q}_0)$ in the n-dimensional space of parameters $(\mathbf{k}, \mathbf{q})$. Let the operator $\mathbf{L}$ be defined by Eq.(2.2) and $\lambda_0 = i\omega_0$ be a double purely imaginary eigenvalue of the operator $\mathbf{L}_0 = \mathbf{L}(0, \mathbf{q}_0)$ with the Jordan chain of length 2, whereas all other eigenvalues are purely imaginary and simple. Since $\mathbf{D}(0)=0$, the non-conservative system corresponding to $\mathbf{k} = 0$, $\mathbf{q} = \mathbf{q}_0$ is circulatory (2.3) and the point $\mathbf{p}_0$ belongs to its stability boundary. Since

$$\mathbf{L}_0 = \mathbf{A}_0 - \omega_0^2 \mathbf{M}_0, \ \frac{\partial \mathbf{L}}{\partial \lambda} = 2i\omega_0 \mathbf{M}_0, \ \frac{\partial^2 \mathbf{L}}{\partial \lambda^2} = 2\mathbf{M}_0, \ \frac{\partial^3 \mathbf{L}}{\partial \lambda^3} = 0,$$

where $\mathbf{A}_0 = \mathbf{A}(\mathbf{q}_0)$ and $\mathbf{M}_0 \equiv \mathbf{M}$, then according to Eqs.(A2),(A3) of Appendix A the eigenvectors $\mathbf{u}_0$, $\mathbf{v}_0$ and associated vectors $\mathbf{u}_1$, $\mathbf{v}_1$ of the double eigenvalue $i\omega_0$ must satisfy the equations

$$(\mathbf{A}_0 - \omega_0^2 \mathbf{M}_0)\mathbf{u}_0 = 0, \quad (\mathbf{A}_0 - \omega_0^2 \mathbf{M}_0)\mathbf{u}_1 = -2i\omega_0 \mathbf{M}_0 \mathbf{u}_0,$$

$$(\mathbf{A}_0^T - \omega_0^2 \mathbf{M}_0^T)\mathbf{v}_0 = 0, \quad (\mathbf{A}_0^T - \omega_0^2 \mathbf{M}_0^T)\mathbf{v}_1 = 2i\omega_0 \mathbf{M}_0^T \mathbf{v}_0. \qquad (2.4)$$

Choose the real vectors $\mathbf{u}_0$, $\mathbf{v}_0$ and purely imaginary vectors $\mathbf{u}_1$, $\mathbf{v}_1$ that satisfy the normalization and orthogonality conditions

$$\mathbf{v}_0^T \frac{\partial \mathbf{L}}{\partial \lambda}\mathbf{u}_1 + \frac{1}{2}\mathbf{v}_0^T \frac{\partial^2 \mathbf{L}}{\partial \lambda^2}\mathbf{u}_0 = 1, \ \mathbf{v}_1^T \frac{\partial \mathbf{L}}{\partial \lambda}\mathbf{u}_1 + \frac{1}{2}\left(\mathbf{v}_1^T \frac{\partial^2 \mathbf{L}}{\partial \lambda^2}\mathbf{u}_0 + \mathbf{v}_0^T \frac{\partial^2 \mathbf{L}}{\partial \lambda^2}\mathbf{u}_1\right) = 0. \qquad (2.5)$$

Consider a smooth perturbation of the vector of parameters $\mathbf{p}(\epsilon)$, $\mathbf{p}(0)=\mathbf{p}_0$ given by Eq.(A4). In the case of the general position the perturbed double eigenvalue is represented by the Newton-Puiseux series $\lambda = i\omega_0 + \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \ldots$ [27]. According to Eqs.(A19)-(A21) and Eq.(A6) we get for the first two coefficients λ_1 and λ_2

$$\lambda_1^2 = -\mathbf{v}_0^T \mathbf{L}_1 \mathbf{u}_0, \quad \lambda_2 = -\frac{1}{2}\left(\mathbf{v}_1^T \mathbf{L}_1 \mathbf{u}_0 + \mathbf{v}_0^T \mathbf{L}_1 \mathbf{u}_1 + \mathbf{v}_0^T \frac{\partial \mathbf{L}_1}{\partial \lambda}\mathbf{u}_0\right), \qquad (2.6)$$

where

$$\mathbf{L}_1 = i\omega_0 \sum_{r=1}^{\mu} \frac{\partial \mathbf{D}}{\partial k_r}\dot{k}_r + \sum_{s=1}^{n-\mu} \frac{\partial \mathbf{A}}{\partial q_s}\dot{q}_s, \quad \frac{\partial \mathbf{L}_1}{\partial \lambda} = \sum_{r=1}^{\mu} \frac{\partial \mathbf{D}}{\partial k_r}\dot{k}_r \qquad (2.7)$$

and dot indicates differentiation with respect to the small parameter ϵ.

For the sake of convenience we introduce the real vectors $\mathbf{f}_k$, $\mathbf{f}_q$, $\mathbf{h}_k$, $\mathbf{h}_q$ with the components

$$f_{k,r} = \mathbf{v}_0^T \frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_0, \quad ih_{k,r} = \mathbf{v}_1^T \frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_0 + \mathbf{v}_0^T \frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_1, \quad r = 1 \ldots \mu, \qquad (2.8)$$

$$f_{q,s} = \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q_s}\mathbf{u}_0, \quad ih_{q,s} = \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial q_s}\mathbf{u}_0 + \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q_s}\mathbf{u}_1 \quad s = 1 \ldots n{-}\mu. \qquad (2.9)$$

Taking into account that according to Eq.(A4) $\Delta\mathbf{k} = \epsilon\dot{\mathbf{k}} + o(\epsilon)$, $\Delta\mathbf{q} = \epsilon\dot{\mathbf{q}} + o(\epsilon)$ and using notation (2.8), (2.9) in Eqs.(2.6), (2.7) we obtain

$$\lambda = i\omega_0 \pm \sqrt{-i\omega_0\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle - \langle \mathbf{f}_q, \Delta\mathbf{q}\rangle} - \frac{1}{2}\left(\langle \mathbf{f}_k {-} \omega_0 \mathbf{h}_k, \Delta\mathbf{k}\rangle + i\langle \mathbf{h}_q, \Delta\mathbf{q}\rangle\right) + \ldots, \quad (2.10)$$

where the angular brackets indicate the inner product of real vectors. Equation (2.10) describes splitting of the double eigenvalue $i\omega_0$ with a change of the vectors $\mathbf{k}$ and $\mathbf{q}$ in the case when the radicand is not zero.

A tangent cone to asymptotic stability boundary. For a fairly small variation of parameters with $\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle \neq 0$ the double non-zero eigenvalue $i\omega_0$ splits into two simple complex eigenvalues, one of them with positive real part. If $\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle = 0$, then for $\langle \mathbf{f}_q, \Delta\mathbf{q}\rangle > 0$ the radical in Eq.(2.10) will be purely imaginary. If, in addition, $\langle \mathbf{h}_k, \Delta\mathbf{k}\rangle < 0$, then for a fairly small variation of parameters the double eigenvalue $i\omega_0$ (as well as $-i\omega_0$) will split into two simple eigenvalues with negative real parts.

However, asymptotic stability depends also on behaviour of the rest of $2m-4$ simple purely imaginary eigenvalues $\pm i\omega_{0,s}$. Choose the real right $\mathbf{u}_{0,s}$ and left $\mathbf{v}_{0,s}$ eigenvectors of these eigenvalues satisfying the normalization conditions

$$\mathbf{v}_{0,s}^T \frac{\partial \mathbf{L}}{\partial \lambda} \mathbf{u}_{0,s} = i.$$

According to Eqs.(A7),(A14) the increments of the eigenvalues $\pm i\omega_{0,s}$ due to change of parameters are described by the expression

$$\lambda_s = \pm\, i\omega_{0,s} \mp i\langle \mathbf{b}_s, \Delta\mathbf{q}\rangle - \omega_{0,s}\langle \mathbf{g}_s, \Delta\mathbf{k}\rangle + o(\|\Delta\mathbf{p}\|^2), \quad s = 1 \ldots m-2,$$

where the real vectors $\mathbf{g}_s$ and $\mathbf{b}_s$ have the components

$$g_{s,r} = \mathbf{v}_{0,s}^T \frac{\partial \mathbf{D}}{\partial k_r} \mathbf{u}_{0,s}, \quad b_{s,j} = \mathbf{v}_{0,s}^T \frac{\partial \mathbf{A}}{\partial q_j} \mathbf{u}_{0,s}, \quad r = 1 \ldots \mu, \quad j = 1 \ldots n-\mu. \qquad (2.11)$$

The condition for $\mathrm{Re}\lambda_s$ to be negative is $\langle \mathbf{g}_s, \Delta\mathbf{k}\rangle > 0$. Therefore, the tangent cone to the domain of asymptotic stability in the space of parameters $\mathbf{k}$ and $\mathbf{q}$ is

$$\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle = 0,\ \langle \mathbf{h}_k, \Delta\mathbf{k}\rangle < 0,\ \langle \mathbf{f}_q, \Delta\mathbf{q}\rangle > 0,\ \langle \mathbf{g}_s, \Delta\mathbf{k}\rangle > 0,\ s = 1 \ldots m-2. \qquad (2.12)$$

One can see from inequalities (2.12) that the domain of asymptotic stability has at the point $\mathbf{p}_0=(0, \mathbf{q}_0)$ a complicated singularity. Indeed, it follows from the works by V.I. Arnold [24] that the asymptotic stability domain in the neighborhood of the point $\mathbf{p}_0$, corresponding to the double non-zero eigenvalue $i\omega_0$ and simple non-zero eigenvalues $i\omega_{0,s}$, s=1, …, $m-2$, is a direct sum of the $(m-2)$-hedral angle and the singularity "Deadlock of an edge", if the number of parameters is sufficiently large. The "Deadlock of an edge" shown in Figure 1 is a generic singularity of the asymptotic stability boundary of a general three-parameter non-conservative system corresponding to the spectrum containing a double purely imaginary eigenvalue (the other eigenvalues having negative real parts) [23, 24].

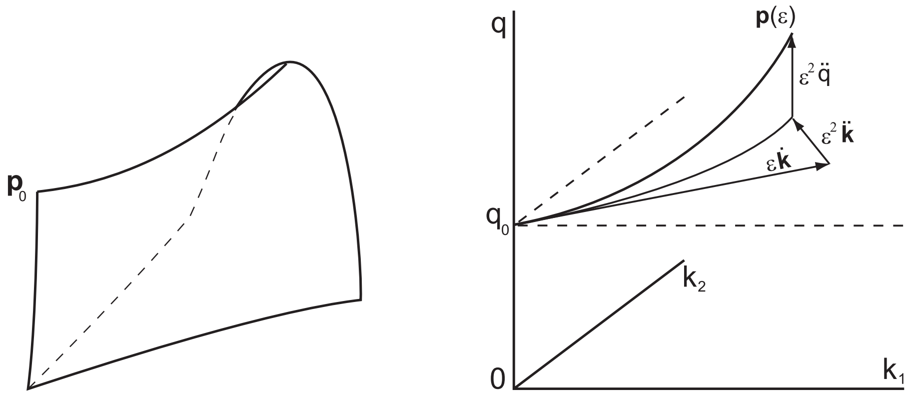

Figure 1. The singularity "Deadlock of an edge" and curve (2.13), (2.14).

Splitting of a double eigenvalue in the degenerate case. To obtain more accurate information on the stability domain in the vicinity of the point $\mathbf{p}_0{=}(0, \mathbf{q}_0)$ we consider a variation of the vector of parameters along a smooth curve

$$\mathbf{p}(\epsilon) = \begin{bmatrix} 0 \\ \mathbf{q}_0 \end{bmatrix} + \epsilon \begin{bmatrix} \dot{\mathbf{k}} \\ 0 \end{bmatrix} + \epsilon^2 \begin{bmatrix} \ddot{\mathbf{k}} \\ \ddot{\mathbf{q}} \end{bmatrix} + o(\epsilon^2). \tag{2.13}$$

Besides, we assume that

$$\langle \mathbf{f}_k, \dot{\mathbf{k}} \rangle = 0. \tag{2.14}$$

Curve (2.13), (2.14) is tangent to the plane $\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle{=}0$ and is orthogonal to the subspace $R^{n-\mu}$ of the parameters $\mathbf{q}$ because $\dot{\mathbf{q}} = 0$, Figure 1.

The radicand in Eq.(2.10) vanishes along curve (2.13), (2.14) and in this degenerate case the double eigenvalue splits linearly with respect to ϵ

$$\lambda = \lambda_0 + \lambda_2 \epsilon + o(\epsilon). \tag{2.15}$$

According to Eqs.(A31), (2.13), and (2.14) the coefficient λ_2 is a root of the quadratic polynomial

$$\lambda_2^2 - \lambda_2 \omega_0 \langle \mathbf{h}_k, \dot{\mathbf{k}} \rangle + \left(\frac{1}{2} \langle \mathbf{f}_q, \ddot{\mathbf{q}} \rangle + \omega_0^2 \langle \mathbf{G}_k \dot{\mathbf{k}}, \dot{\mathbf{k}} \rangle \right) + i\omega_0 \left(\frac{1}{2} \langle \mathbf{f}_k, \ddot{\mathbf{k}} \rangle + \langle \mathbf{H}_k \dot{\mathbf{k}}, \dot{\mathbf{k}} \rangle \right) = 0. \tag{2.16}$$

The vectors $\mathbf{f}_k$, $\mathbf{f}_q$, $\mathbf{h}_k$ are defined by Eqs.(2.8),(2.9), the real $\mu \times \mu$-matrix $\mathbf{H}_k$ has the components

$$H_{r,s} = \frac{1}{2} \mathbf{v}_0^T \frac{\partial^2 \mathbf{D}}{\partial k_r \partial k_s} \mathbf{u}_0,$$

and the real $\mu \times \mu$-matrix $\mathbf{G}_k$ is defined by the following expression

$$\langle \mathbf{G}_k \dot{\mathbf{k}}, \dot{\mathbf{k}} \rangle = \sum_{r=1}^{\mu} \dot{k}_r \mathbf{v}_0^T \mathbf{G}_0 \left(\sum_{s=1}^{\mu} \dot{k}_s \frac{\partial \mathbf{D}}{\partial k_s} \mathbf{u}_0 \right), \tag{2.17}$$

where $\mathbf{G}_0$ is the operator inverse to $\mathbf{L}_0 = \mathbf{A}_0 - \omega_0^2\mathbf{M}_0$.

Taking into account explicit expression (2.13) for the curve $\mathbf{p}(\epsilon)$ and Eq.(2.15) we rewrite Eq.(2.16) in the form

$$\Delta\lambda^2 - \Delta\lambda\omega_0\langle\mathbf{h}_k, \Delta\mathbf{k}\rangle + \langle\mathbf{f}_q, \Delta\mathbf{q}\rangle + \omega_0^2\langle\mathbf{G}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle + i\omega_0\left(\langle\mathbf{f}_k, \Delta\mathbf{k}\rangle + \langle\mathbf{H}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle\right) = 0. \tag{2.18}$$

According to the Bilharz criterion [28] the necessary and sufficient conditions for a root of complex polynomial (2.18) to have a negative real part are (see Eq.(C2))

$$\langle\mathbf{f}_q, \Delta\mathbf{q}\rangle > \frac{(\langle\mathbf{f}_k, \Delta\mathbf{k}\rangle + \langle\mathbf{H}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle)^2}{\langle\mathbf{h}_k, \Delta\mathbf{k}\rangle^2} - \omega_0^2\langle\mathbf{G}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle, \tag{2.19}$$

$$\omega_0\langle\mathbf{h}_k, \Delta\mathbf{k}\rangle < 0. \tag{2.20}$$

Inequalities (2.19), (2.20) give approximation (more accurate than the tangent cone) of the domain in the parameter space, where the double eigenvalue $i\omega_0$ splits into two complex eigenvalues with negative real parts.

The critical flutter load and the boundary of asymptotic stability. Consider an important case when the matrix $\mathbf{A}$ is a function of only one parameter q and the matrix $\mathbf{D}$ depends on the vector $\mathbf{k}$. Then, Eq.(2.19) allows to find a critical value of the parameter q, at which splitting of the double eigenvalue $i\omega_0$ causes asymptotic instability (flutter), as a function of the vector $\mathbf{k}$

$$q_{cr} = q_0 + \frac{(\langle\mathbf{f}_k, \Delta\mathbf{k}\rangle + \langle\mathbf{H}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle)^2}{f_q\langle\mathbf{h}_k, \Delta\mathbf{k}\rangle^2} - \frac{\omega_0^2}{f_q}\langle\mathbf{G}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle. \tag{2.21}$$

Restricting our consideration by the case when

$$\{\mathbf{k} : q > q_{cr}(\mathbf{k})\} \subset \{\mathbf{k} : \omega_0\langle\mathbf{h}_k, \mathbf{k}\rangle < 0, \quad \langle\mathbf{g}_s, \mathbf{k}\rangle > 0, \quad s = 1 \dots m-2\} \tag{2.22}$$

meaning that all the simple eigenvalues $\pm i\omega_{0,s}$, $s = 1, \dots, m-2$ move to the left-hand half of the complex plane we conclude that the surface $q_{cr}(\mathbf{k})$ approximated by Eq.(2.21) under the constraint (2.20) is the boundary of the domain of asymptotic stability.

Level sets of function (2.21) are stability boundaries in the space of parameters $\mathbf{k}=(k_1, \dots, k_\mu)$. The level set $q_{cr}=q_0$, where q_0 is the critical value of the parameter q for the unperturbed circulatory system, is given by the expression

$$\langle\mathbf{H}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle = -\langle\mathbf{f}_k, \Delta\mathbf{k}\rangle \pm \omega_0\langle\mathbf{h}_k, \Delta\mathbf{k}\rangle\sqrt{\langle\mathbf{G}_k\Delta\mathbf{k}, \Delta\mathbf{k}\rangle}. \tag{2.23}$$

If there are only two parameters $\mathbf{k} = (k_1, k_2)$, then (2.21) is the equation of the surface known as the Whitney umbrella [24]. The level curves of this surface for

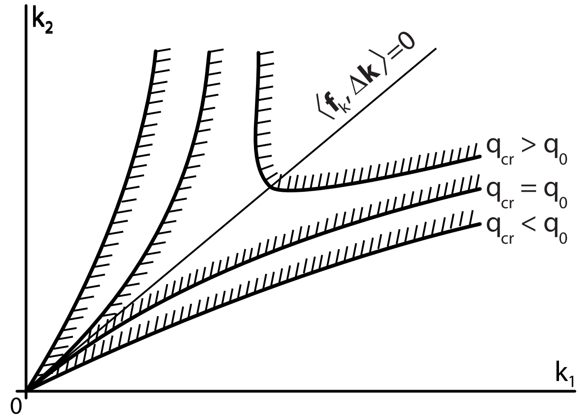

Figure 2. Level curves of the function $q_{cr}(k_1, k_2)$ for $f_q < 0$. The asymptotic stability domains are hatched.

$f_q < 0$ are given in Figure 2. One can see that for $q_{cr}{<}q_0$ the asymptotic stability domain has an angle in the plane of parameters (k_1, k_2). At $q_{cr}{>}q_0$ the asymptotic stability domain is far away from the origin, Figure 2, and one needs to apply sufficiently large velocity-dependent forces to stabilize the system. At $q_{cr}{=}q_0$ the asymptotic stability domain has a degenerate singularity, the "cusp", at the origin of the plane of parameters. From Eq.(2.23) we find equivalent approximations of the cusp expressing $\Delta k_1{=}k_1$ as a function of $\Delta k_2{=}k_2$ and vice versa

$$k_1 = -\frac{f_{k,2}}{f_{k,1}}k_2 - \frac{\mathbf{f}_k^T\mathbf{H}_k^{\dagger}\mathbf{f}_k \pm \omega_0(h_{k,1}f_{k,2} - h_{k,2}f_{k,1})\sqrt{\mathbf{f}_k^T\mathbf{G}_k^{\dagger}\mathbf{f}_k}}{f_{k,1}^3}k_2^2 + o(k_2^2), \qquad (2.24)$$

$$k_2 = -\frac{f_{k,1}}{f_{k,2}}k_1 - \frac{\mathbf{f}_k^T\mathbf{H}_k^{\dagger}\mathbf{f}_k \pm \omega_0(h_{k,1}f_{k,2} - h_{k,2}f_{k,1})\sqrt{\mathbf{f}_k^T\mathbf{G}_k^{\dagger}\mathbf{f}_k}}{f_{k,2}^3}k_1^2 + o(k_1^2), \qquad (2.25)$$

where the symbol of dagger indicates the matrices adjoint to $\mathbf{H}_k$ and $\mathbf{G}_k$ [29]

$$\mathbf{H}_k^{\dagger} = \begin{bmatrix} H_{k,22} & -H_{k,12} \\ -H_{k,21} & H_{k,11} \end{bmatrix}, \quad \mathbf{G}_k^{\dagger} = \begin{bmatrix} G_{k,22} & -G_{k,12} \\ -G_{k,21} & G_{k,11} \end{bmatrix}.$$

Movement of eigenvalues in the complex plane. Assume that the matrix $\mathbf{A}$ is a function of only one parameter q and the matrix $\mathbf{D}$ depends on the vector $\mathbf{k}$. Substitution of $\Delta\lambda = \mathrm{Re}\lambda + i(\mathrm{Im}\lambda - \omega_0)$ into Eq.(2.18) and separation of real and imaginary parts yields

$$(\mathrm{Im}\lambda{-}\omega_0 + \mathrm{Re}\lambda + a/2)^2 - (\mathrm{Im}\lambda{-}\omega_0 - \mathrm{Re}\lambda - a/2)^2 = -2d, \qquad (2.26)$$

$$\left(\mathrm{Re}\lambda{+}\frac{a}{2}\right)^4 + \left(c{-}\frac{a^2}{4}\right)\left(\mathrm{Re}\lambda{+}\frac{a}{2}\right)^2 {=}\frac{d^2}{4}, \qquad (2.27)$$

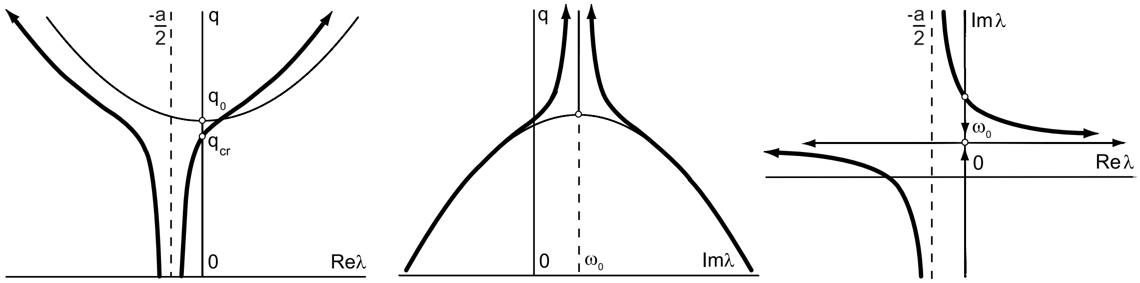

Figure 3. Movement of eigenvalues of an unperturbed circulatory system (thin lines) and the system with small velocity-dependent forces (thick lines) for $d \neq 0$.

$$(\mathrm{Im}\lambda - \omega_0)^4 - \left(c - \frac{a^2}{4}\right)(\mathrm{Im}\lambda - \omega_0)^2 = \frac{d^2}{4}, \tag{2.28}$$

where

$$a = -\omega_0 \langle \mathbf{h}_k, \Delta\mathbf{k}\rangle, \quad c = f_q \Delta q + \omega_0^2 \langle \mathbf{G}_k \Delta\mathbf{k}, \Delta\mathbf{k}\rangle, \quad d = \omega_0 \left(\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle + \langle \mathbf{H}_k \Delta\mathbf{k}, \Delta\mathbf{k}\rangle\right).$$

Equations (2.26)–(2.28) describe the movement of eigenvalues λ in the complex plane with a small variation of the stiffness matrix $\mathbf{A}(q)$ and the matrix of velocity-dependent forces $\mathbf{D}(\mathbf{k})$. Indeed, if $\Delta\mathbf{k}=0$, then the system is circulatory and with a change of the parameter q two simple purely imaginary eigenvalues move along the imaginary axis, merge at $q=q_0$ and then diverge in the direction perpendicular to the imaginary axis with the origination of a pair of simple complex eigenvalues (flutter). Since at $\Delta\mathbf{k}=0$ the quantities $a=0$, $c=f_q\Delta q$, $d=0$ the eigenvalues move in accordance with Eqs.(2.26)–(2.28), which take the form

$$\begin{aligned} &\mathrm{Re}\lambda = \pm\sqrt{-f_q(q-q_0)}, \quad \mathrm{Im}\lambda = \omega_0, \quad f_q(q-q_0) \le 0, \\ &\mathrm{Re}\lambda = 0, \quad \mathrm{Im}\lambda = \omega_0 \pm \sqrt{f_q(q-q_0)}, \quad f_q(q-q_0) \ge 0. \end{aligned} \tag{2.29}$$

The movement of eigenvalues of a circulatory system with a change of the parameter q is shown in Figure 3 by thin lines for $f_q < 0$. Such a behaviour of eigenvalues is known as the *strong interaction* [18, 25].

If $\Delta\mathbf{k}\neq 0$ and $d\neq 0$, then the velocity-dependent forces destroy the strong interaction of eigenvalues as shown in Figure 3. For the fixed $\Delta\mathbf{k}$ the eigenvalues move with a change of the parameter q along the branches of hyperbola given by Eq.(2.26) in the complex plane. This hyperbola has two asymptotes $\mathrm{Re}\lambda = -a/2$ and $\mathrm{Im}\lambda = \omega_0$. If $a < 0$, then one of the two eigenvalues is always in the right-hand half of the complex plane making the system asymptotically unstable. Thus, $a > 0$ is a necessary condition of asymptotic stability, which coincides with inequality (2.20) obtained with the use of the Bilharz criterion.

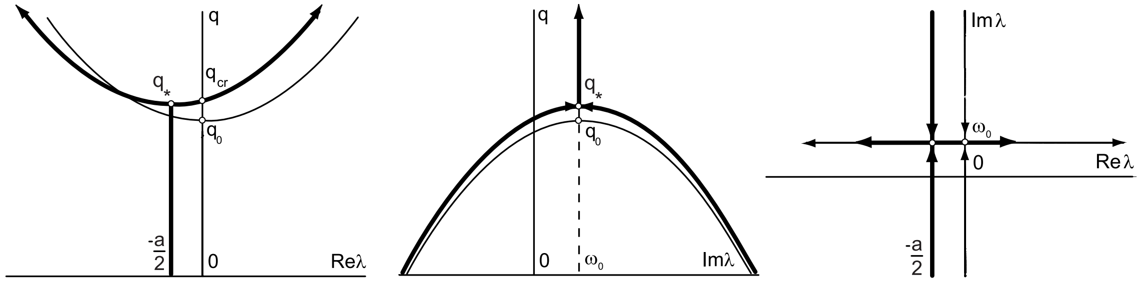

Figure 4. Movement of eigenvalues of an unperturbed circulatory system (thin lines) and the system with small velocity-dependent forces (thick lines) for $d = 0$.

Behaviour of real and imaginary parts of eigenvalues λ is governed by Eqs.(2.27), (2.28). Characteristic curves $\mathrm{Re}\lambda(q)$ and $\mathrm{Im}\lambda(q)$ are shown in Figure 3 by thick lines. The value of the parameter q at which $\mathrm{Re}\lambda = 0$ follows from Eq.(2.27) in the form $ca^2 = d^2$. It is not surprising that the obtained condition exactly coincides with Eq.(2.21) for the critical load q_{cr}. One can see that the velocity-dependent forces shift and split the characteristic curves of a circulatory system as shown in Figure 3. This qualitative fact was known in the literature, see for example [3, 11, 30]. Equations (2.26)–(2.28) that describe analytically the behaviour of eigenvalues of the circulatory system perturbed by velocity-dependent forces as well as Eq.(2.21) for the critical load seem to appear first in the present paper.

In the case d=0 the introduction of small velocity-dependent forces ($\Delta\mathbf{k}\neq 0$) does not lead to the destruction of the strong interaction of eigenvalues. The complex eigenvalues λ with $\mathrm{Re}\lambda = -a/2$ interact strongly at q=q_*, where

$$q_* = q_0 + \omega_0^2 \frac{\langle \mathbf{h}_k, \Delta\mathbf{k}\rangle^2 - 4\langle \mathbf{G}_k \Delta\mathbf{k}, \Delta\mathbf{k}\rangle}{4 f_q} \tag{2.30}$$

in accordance with Eqs.(2.26)–(2.28). Then, with a change of the parameter q the double eigenvalue $\lambda_* = -a/2 + i\omega_0$ splits into two simple complex eigenvalues, one of which crosses the imaginary axis at $q = q_{cr}$ given by Eq.(2.21) as shown in Figure 4 for $a > 0$.

Change in the static instability mechanism due to perturbation by small velocity-dependent forces. Assume that at the point $\mathbf{p}_0$=$(0, \mathbf{q}_0)$ corresponding to a non-perturbed circulatory system there is a double eigenvalue $\lambda_0 = 0$ with the Jordan chain of length 2, all other eigenvalues being purely imaginary and simple. Such points, which can be denoted by the symbol "0^2" [24], constitute smooth parts of the boundary between the stability and static instability (divergence) domains of a circulatory system [25]. However, a non-conservative system

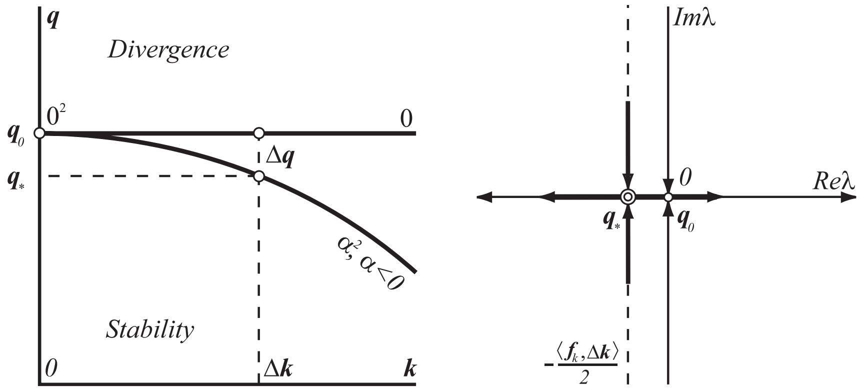

Figure 5. The bifurcation diagram and the splitting of the double zero eigenvalue in the unperturbed (thin lines) and perturbed (thick lines) cases.

with damping and gyroscopic forces loses stability by divergence when a simple negative real eigenvalue passes through the origin "0" to the right-hand side of the complex plane. Therefore, velocity-dependent forces change the static instability mechanism.

It is known [24] that for a general non-conservative system the points of the type "0" corresponding to simple zero eigenvalues and the points of the type "α^2" corresponding to double real non-zero eigenvalues with the Jordan chain of length 2 constitute smooth hypersurfaces in the parameter space, which are tangent to each other at the points of the type "0^2". The bifurcation diagram in the vicinity of the point $\mathbf{p}_0=(0,\mathbf{q}_0)$ is shown in Figure 5. Note that the hypersurface of the type "0" does not change due to variation of parameters $\mathbf{k}$ because $\det(\lambda^2+\mathbf{D}(\mathbf{k})\lambda+\mathbf{A}(\mathbf{q}))_{\lambda=0}=\det\mathbf{A}(\mathbf{q})$. This hypersurface is orthogonal to the subspace $R^{n-\mu}$ of the parameters $\mathbf{q}$ as it is shown in Figure 5.

Thus, to understand how the static instability mechanism changes we should investigate the splitting of the double zero eigenvalue along curves of the type (2.13). Taking into account Eqs.(2.5), (2.7)–(2.9), and (2.13) and substituting $\omega_0=0$ into Eq.(A31) we get

$$\lambda^2+\lambda\langle\mathbf{f}_k,\Delta\mathbf{k}\rangle+\langle\mathbf{f}_q,\Delta\mathbf{q}\rangle=0. \tag{2.31}$$

If $\Delta\mathbf{k}=0$, then the non-conservative system is circulatory. Its spectrum contains the double eigenvalue $\lambda=0$ at the point $\mathbf{q}=\mathbf{q}_0$ of the boundary between the stability and divergence domains. With a change of the vector of parameters $\Delta\mathbf{q}=\mathbf{q}-\mathbf{q}_0\neq0$ the double zero eigenvalue bifurcates passing through the divergence boundary according to the formula

$$\lambda=\pm\sqrt{-\langle\mathbf{f}_q,\Delta\mathbf{q}\rangle}. \tag{2.32}$$

This is the typical mechanism of the static instability in circulatory systems [25].

If the damping is introduced ($\Delta\mathbf{k} \neq 0$) then the double zero eigenvalue splits with a change of parameters in the following way

$$\lambda_{1,2} = -\frac{\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle}{2} \pm \frac{1}{2}\sqrt{\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle^2 - 4\langle \mathbf{f}_q, \Delta\mathbf{q}\rangle}. \tag{2.33}$$

If $\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle > 0$ then at $\mathbf{q} = \mathbf{q}_0$ the double zero eigenvalue splits into a simple zero eigenvalue and a negative real eigenvalue

$$\lambda_1 = 0, \quad \lambda_2 = -\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle. \tag{2.34}$$

This shows that the hypersurface "0" remains the same while the static instability occurs due to passing of the simple eigenvalue through the origin of the complex plane. If $\langle \mathbf{f}_q, \Delta\mathbf{q}\rangle > 0$, then on the surface approximated by the expression

$$\langle \mathbf{f}_q, \Delta\mathbf{q}\rangle = \frac{\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle^2}{4} \tag{2.35}$$

the eigenvalue remains double

$$\lambda_1 = \lambda_2 = -\frac{\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle}{2}. \tag{2.36}$$

If $\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle > 0$, then for a fixed vector $\Delta\mathbf{k}$ the double zero eigenvalue rebounds from the imaginary axis and becomes double negative according to Eq.(2.36). Due to change of the vector $\mathbf{q}$ two complex-conjugate eigenvalues collide at the surface (2.35) when $\mathbf{q} = \mathbf{q}_*$ with the origination of the double negative eigenvalue (2.36). At $\mathbf{q} = \mathbf{q}_0$ one of these simple eigenvalues becomes zero while another one remains negative (2.34). This process is shown in Figure 5. In the case when

$$\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle = 0 \tag{2.37}$$

the double zero eigenvalue does not rebound from the imaginary axis due to variation of the vector $\mathbf{k}$.

3 The Herrmann-Jong pendulum

As an example we consider a double pendulum composed of two rigid weightless bars of equal length l, which carry concentrated masses $m_1{=}2m$, $m_2{=}m$. The generalized coordinates φ_1 and φ_2 are assumed to be small. A load Q is applied at the free end at an angle $\eta\varphi_2$, as shown in Figure 6. At the hinges, the restoring moments

$c\varphi_1+b_1 d\varphi_1/dt$ and $c(\varphi_2-\varphi_1)+b_2(d\varphi_2/dt-d\varphi_1/dt)$ are induced. The linear equations of motion are

$$3ml^2\frac{d^2\varphi_1}{dt^2}+(b_1+b_2)\frac{d\varphi_1}{dt}-(Ql-2c)\varphi_1+ml^2\frac{d^2\varphi_2}{dt^2}-b_2\frac{d\varphi_2}{dt}+(\eta Ql-c)\varphi_2=0,$$

$$ml^2\frac{d^2\varphi_1}{dt^2}-b_2\frac{d\varphi_1}{dt}-c\varphi_1+ml^2\frac{d^2\varphi_2}{dt^2}+b_2\frac{d\varphi_2}{dt}-((1-\eta)Ql-c)\varphi_2=0, \quad (3.1)$$

where t indicates time, b_1 and b_2 are the damping coefficients and c characterizes the elastic properties of the hinges [9]. After introduction of the dimensionless quantities

$$q=\frac{Ql}{c}, \quad k_1=\frac{b_1}{\sqrt{cml^2}}, \quad k_2=\frac{b_2}{\sqrt{cml^2}}, \quad \tau=t\sqrt{\frac{c}{ml^2}}$$

we arrive at the equation in the form (2.1), where

$$\mathbf{M}=\left[\begin{array}{cc}3&1\\1&1\end{array}\right], \quad \mathbf{D}=\left[\begin{array}{cc}k_1+k_2&-k_2\\-k_2&k_2\end{array}\right], \quad \mathbf{A}=\left[\begin{array}{cc}2-q&\eta q-1\\-1&1-(1-\eta)q\end{array}\right]. \quad (3.2)$$

Calculating $\det(\mathbf{M}\lambda^2+\mathbf{D}\lambda+\mathbf{A})$ and equating it to zero we obtain the characteristic equation

$$2\lambda^4+(6k_2+k_1)\lambda^3+(k_1k_2+2\eta q+7-4q)\lambda^2+((2k_2q+k_1q)(\eta-1)+k_2+k_1)\lambda+$$

$$+(3q-q^2)(\eta-1)+1=0. \quad (3.3)$$

Stability boundaries of the undamped and damped systems. In the case when damping is absent the flutter boundary is defined by the double purely imaginary eigenvalues with the Jordan chains of length 2. Since for k_1=k_2=0 equation (3.3) becomes biquadratic, then the flutter boundary of the circulatory system can be found by equating the corresponding discriminant to zero

$$q=\frac{8-\eta\pm\sqrt{-18+66\eta-40\eta^2}}{4-4\eta+2\eta^2}. \quad (3.4)$$

The divergence boundary of the circulatory system in the plane of parameters (η, q) follows from Eq.(3.3) after equating its free term to zero

$$\eta=1-\frac{1}{q(3-q)}. \quad (3.5)$$

Almost all points of the divergence boundary of the circulatory system correspond to the double zero eigenvalues of the type "0^2". The boundaries between the stability (S), flutter (F), and divergence (D) domains of the circulatory system on the plane of the parameters η and q are shown in Figure 6 by thick lines.

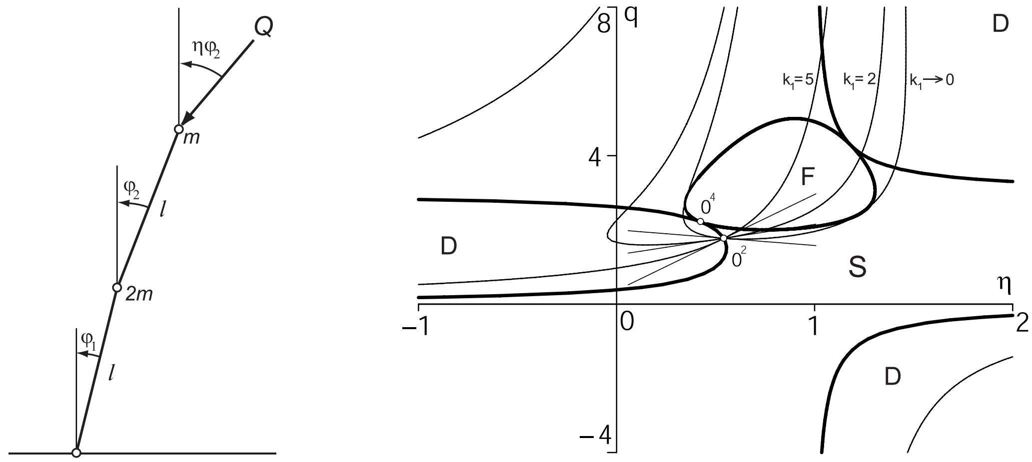

Figure 6. The Herrmann-Jong pendulum: Stability diagrams for k_1=0, k_2=0 (thick lines) and in the case when $k_2 = 3/10\ k_1$ for $k_1 \longrightarrow 0$, k_1=2, and k_1=5 (thin lines).

One can see that Eq.(3.3) possesses quadruple zero eigenvalue "0^4" if all the coefficients of the characteristic polynomial are zero. The point in the plane of parameters corresponding to this eigenvalue has the coordinates

$$k_1 = k_2 = 0, \quad \eta = \frac{61}{76} - \frac{7}{76}\sqrt{17}, \quad q = \frac{13}{4} - \frac{1}{4}\sqrt{17} \tag{3.6}$$

being common both for the divergence and flutter boundaries, Figure 6. At this point the stability boundary of the circulatory system has a generic singularity – the "cusp" [26].

The Routh-Hurwitz criterion applied to Eq.(3.3) gives the asymptotic stability boundary for the non-conservative system with dissipation

$$R(\eta, q, k_1, k_2) \equiv q^2(\eta - 1)(4k_2^2(4\eta - 1) + 4k_1k_2(2\eta - 3) - k_1^2)+$$
$$+q(\eta - 1)(8k_1k_2(k_1^2 + 8k_1k_2 + 12k_2^2) + 2k_1^2 + 22k_1k_2 - 20k_2^2)+$$
$$+(k_1k_2 - 2q)(k_1^2 + 7k_1k_2 + 6k_2^2) + 4k_1^2 + 33k_1k_2 + 4k_2^2 = 0. \tag{3.7}$$

These boundaries for different values of the dissipation parameters k_1, k_2 are shown in Figure 6 on the plane of parameters η and q by thin lines.

After introduction of small damping the divergence boundary (3.5) remains the same but most of its points correspond to the simple zero eigenvalues. However, some isolated points of the divergence boundary still correspond to the double zero eigenvalue "0^2". The double zero is a root of characteristic polynomial (3.3) for $k_1 \neq 0$, $k_2 \neq 0$ if the free term and the coefficient at λ vanish simultaneously. The corresponding point of the divergence boundary has the coordinates

$$q = \frac{k_2 + 2k_1}{k_1 + k_2}, \quad \eta = \frac{k_1^2 + k_2^2 + 3k_1k_2}{2(k_1^2 + k_2^2) + 5k_1k_2}. \tag{3.8}$$

Note that asymptotic stability boundary (3.7) passes through the point (3.8) for different values of the parameters k_1, k_2 as shown in Figure 6.

Splitting of the double zero and coordinates of the point "0^2"**.** As it was shown in Section 2 the double zero eigenvalue in general rebounds from the imaginary axis due to small damping. The condition for the double zero eigenvalue to remain on the imaginary axis is Eq.(2.37). Let us find the point "0^2" with the coordinates given by Eq.(3.8) by means of the non-rebound condition given by Eq.(2.37).

From the equations $\mathbf{A}_0\mathbf{u}_0 = 0$ and $\mathbf{A}_0^T\mathbf{v}_0 = 0$ we obtain the left and right eigenvectors of a zero eigenvalue

$$\mathbf{u}_0 = C_1\begin{bmatrix} 1 \\ (3-q)/(2-q) \end{bmatrix}, \quad \mathbf{v}_0 = C_2\begin{bmatrix} 1 \\ 2-q \end{bmatrix}, \tag{3.9}$$

where C_1 and C_2 are arbitrary coefficients and the matrix $\mathbf{A}_0$

$$\mathbf{A}_0 = \begin{bmatrix} 2-q & -(2-q)^2/(3-q) \\ -1 & (2-q)/(3-q) \end{bmatrix}. \tag{3.10}$$

is the matrix $\mathbf{A}$ given by Eq.(3.2) evaluated at the points of the divergence boundary (3.5). Normalization condition $(\mathbf{M}\mathbf{u}_0, \mathbf{v}_0)$=1 gives the relationship between C_1, C_2

$$C_1C_2 = \frac{2-q}{2q^2-13q+19}. \tag{3.11}$$

With these ingredients one can find the vector $\mathbf{f}_k = (f_{k,1}, f_{k,2})$

$$f_{k,1} \equiv \mathbf{v}_0^T\frac{\partial \mathbf{D}}{\partial k_1}\mathbf{u}_0 = C_1C_2, \quad f_{k,2} \equiv \mathbf{v}_0^T\frac{\partial \mathbf{D}}{\partial k_2}\mathbf{u}_0 = C_1C_2\frac{1-q}{2-q}, \tag{3.12}$$

and the vector $\mathbf{f}_q = (f_{q,1}, f_{q,2})$ with the components

$$f_{q,1} \equiv \mathbf{v}_0^T\frac{\partial \mathbf{A}}{\partial \eta}\mathbf{u}_0 = \frac{q(3-q)^2}{2q^2-13q+19}, \quad f_{q,2} \equiv \mathbf{v}_0^T\frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_0 = \frac{2q-3}{q(2q^2-13q+19)}. \tag{3.13}$$

Therefore, condition (2.37) takes the form

$$\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle = \frac{2-q}{2q^2-13q+19}\left(k_1 + \frac{1-q}{2-q}k_2\right) = 0 \tag{3.14}$$

From this equation we find the ordinate of the point "0^2"

$$q = \frac{k_2+2k_1}{k_1+k_2},$$

which completely coincides with the ordinate given by the first of Eqs.(3.8). Substitution of this ordinate into equation of the divergence boundary (3.5) yields the second coordinate, which evidently coincides with the second of Eqs.(3.8). From the other point of view at a point of the divergence boundary with the ordinate q the double zero eigenvalue does not rebound from the imaginary axis if

$$\frac{k_1}{k_2} = \frac{q-1}{2-q}. \tag{3.15}$$

Linear approximation of the flutter boundary in the vicinity of 0^2. Thus, for the fixed ratio k_1/k_2 there exists a point (3.8) on the divergence boundary, where the zero eigenvalue remains double and the flutter boundary on the plane (η, q) passes through this point as shown in Figure 6. With a change of one of the parameters k_1 or k_2 the flutter boundary rotates around the point "0^2". This effect is known also for the continuous systems [11]. To understand how sensitive is the flutter boundary to the changes in the dissipation parameters it is interesting to find the linear approximation of the boundary in the vicinity of the point (3.8).

Calculating the first partial derivatives of the function $R(\eta, q, k_1, k_2)$ given by Eq.(3.7) with respect to the parameters η and q we get the coefficients

$$a = \left(12\beta^2 + 20\beta - 3 + \frac{5}{1+\beta}\right)\beta k_1^2 + \left(-24\beta - 10 + \frac{8+23\beta}{(1+\beta)^2}\right)\beta, \quad \beta = \frac{k_2}{k_1},$$

$$b = \left(-6\beta^2 - \beta - 6 + \frac{11+22\beta}{2+5\beta+2\beta^2}\right)\beta k_1^2 - 6\beta^2 - 28\beta + 27 - \frac{105\beta+56}{2+5\beta+2\beta^2}, \tag{3.16}$$

of the linear approximation $a\Delta\eta + b\Delta q = 0$ of the asymptotic stability boundary in the neighborhood of the point "0^2", which has the coordinates given by Eq.(3.8).

For example, at the point of the divergence boundary with the coordinates

$$\eta_0 = \frac{199}{368}, \quad q_0 = \frac{23}{13}, \tag{3.17}$$

the double zero eigenvalue does not rebound from the imaginary axis if according to equation (3.15)

$$k_2 = \frac{3}{10}k_1. \tag{3.18}$$

Substituting ratio (3.18) into Eqs.(3.16) we get the linear approximation of the flutter boundary near the point (3.17)

$$\left(\frac{3864}{1625}k_1^2 - \frac{10626}{4225}\right)\left(\eta - \frac{199}{368}\right) - \left(\frac{3549}{5750}k_1^2 + \frac{26299}{4600}\right)\left(q - \frac{23}{13}\right) = 0. \tag{3.19}$$

The flutter boundaries with their linear approximations given by Eq.(3.19) for the different values of parameters k_1 and k_2 are shown in Figure 6.

Rebound of the double zero eigenvalue from the imaginary axis. Consider a point of the divergence boundary with the coordinates

$$\eta_0 = 0, \quad q_0 = \frac{3-\sqrt{5}}{2} \tag{3.20}$$

and fix the coordinate η_0. According to Eqs.(3.12), (3.13) the vector $\mathbf{f}_k$ and the quantity $f_{q,2}$ evaluated at the point (3.20) are

$$f_{k,1} = \frac{11-3\sqrt{5}}{38}, \quad f_{k,2} = \frac{11-3\sqrt{5}}{38}\frac{3-\sqrt{5}}{2}, \quad f_{q,2} = -\frac{11-3\sqrt{5}}{76}(5+\sqrt{5}). \tag{3.21}$$

Due to addition of the damping parameters the double zero eigenvalue rebounds from the imaginary axis at the value of the parameter $q = q_*$, which follows from Eq.(2.35) after the substitution of Eqs.(3.20), (3.21)

$$q_* = \frac{3-\sqrt{5}}{2} - \left(\frac{11}{38} - \frac{49}{380}\sqrt{5}\right)\left(k_2 + \frac{3+\sqrt{5}}{2}k_1\right)^2 < q_0 \tag{3.22}$$

and becomes double non-zero according to the formula

$$\lambda_1 = \lambda_2 = -\frac{\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle}{2} = -\frac{12-5\sqrt{5}}{38}\left(k_2 + \frac{3+\sqrt{5}}{2}k_1\right). \tag{3.23}$$

On the divergence boundary $q = q_0$ we have

$$\lambda_1 = 0, \quad \lambda_2 = -\langle \mathbf{f}_k, \Delta\mathbf{k}\rangle = -\frac{12-5\sqrt{5}}{19}\left(k_2 + \frac{3+\sqrt{5}}{2}k_1\right). \tag{3.24}$$

Compare the theoretical results with the numerical computations for the point

$$\eta_0 = 0, \quad q_0 = \frac{3}{2} - \frac{\sqrt{5}}{2} \simeq 0.38196601, \quad k_1 = 0.1, \quad k_2 = 0.1. \tag{3.25}$$

From the characteristic equation (3.3) we find the coordinate of the point q^* with the double non-zero eigenvalue

$$q^* = 0.38181696, \quad \lambda_1 = \lambda_2 = -0.00780441. \tag{3.26}$$

Approximate formulae (3.22), (3.23) give

$$q^* = 0.38181696, \quad \lambda_1 = \lambda_2 = -0.00780410. \tag{3.27}$$

One can conclude looking at expressions (3.26) and (3.27) that the theory developed is in a very good agreement with the numerical results.

Approximation of the domain of asymptotic stabilization. Consider the case when $\eta = 1$ and the force q acts on the free end of the pendulum being always tangent to the bar (if, additionally k_1=k_2, we get the pendulum considered by H. Ziegler [2]). Then, the characteristic polynomial (3.3) takes the form

$$2\lambda^4 + (6k_2 + k_1)\lambda^3 + (k_1k_2 - 2q + 7)\lambda^2 + \lambda(k_1 + k_2) + 1 = 0. \qquad (3.28)$$

If the damping is absent, then the system is marginally stable for the loads $q < q_0$, where the critical load q_0 corresponds to the double eigenvalue $\lambda_0 = \pm i\omega_0$

$$\omega_0 = 2^{-1/4}, \quad q_0 = \frac{7}{2} - \sqrt{2}. \qquad (3.29)$$

Substituting η=1 in Eq.(3.7) we find the critical load of the damped system [8]

$$q_{cr} = \frac{4k_1^2 + 33k_1k_2 + 4k_2^2}{2(k_1^2 + 7k_1k_2 + 6k_2^2)} + \frac{1}{2}k_1k_2. \qquad (3.30)$$

The domain of asymptotic stability is defined by the inequality $q < q_{cr}$.

Now we apply the theory developed in Section 2 to approximate the asymptotic stability domain. First of all we should find the right and left Jordan chains of the double eigenvalue λ_0=$i\omega_0$ at the load q_0 given by Eq.(3.29). Solution of Eqs.(2.4) and the second of Eqs.(A23) yields

$$\mathbf{u}_0=\begin{bmatrix} 5\sqrt{2}-6 \\ 3\sqrt{2}+2 \end{bmatrix}, \ \mathbf{u}_1=8i\omega_0\begin{bmatrix} 5\sqrt{2}-6 \\ 0 \end{bmatrix}, \ \mathbf{v}_0=\frac{-1}{112}\begin{bmatrix} \sqrt{2}+4 \\ -7 \end{bmatrix}, \ \mathbf{v}_1=\frac{i\omega_0}{112}\begin{bmatrix} 8\sqrt{2}-10 \\ 36-19\sqrt{2} \end{bmatrix},$$

$$\mathbf{G}_0\left(\sum_{s=1}^{2}\dot{k}_s\frac{\partial \mathbf{D}}{\partial k_s}\mathbf{u}_0\right) = 4\dot{k}_2(3\sqrt{2} - 5, 0). \qquad (3.31)$$

Evaluation of the vectors $\mathbf{f}_k$, $\mathbf{f}_q$ and $\mathbf{h}_k$, $\mathbf{h}_q$ as well as the matrices $\mathbf{H}_k$ and $\mathbf{G}_k$ given by Eqs.(2.8),(2.9), and Eq.(2.17) yields:

$$\mathbf{f}_k = \frac{1}{8}\begin{bmatrix} 1-\sqrt{2} \\ 6-\sqrt{2} \end{bmatrix}, \ f_q = -\frac{1}{4}; \ \mathbf{h}_k = -\frac{1}{8\omega_0}\begin{bmatrix} 1+\sqrt{2} \\ 6+\sqrt{2} \end{bmatrix}, \ h_q = \omega_0\frac{\sqrt{2}}{4};$$

$$\mathbf{H}_k = 0, \quad \mathbf{G}_k = \frac{1}{8}\begin{bmatrix} 0 & 2-\sqrt{2} \\ 2-\sqrt{2} & 14-8\sqrt{2} \end{bmatrix}. \qquad (3.32)$$

Substitution of the vectors and matrices given by Eq.(3.32) into Eq.(2.21) yields

$$q_{cr}=q_0-2\sqrt{2}\left(\frac{k_1(\sqrt{2}-1)+k_2(\sqrt{2}-6)}{k_1(\sqrt{2}+1)+k_2(\sqrt{2}+6)}\right)^2+2\sqrt{2}\left(\left(\frac{7}{4}-\sqrt{2}\right)k_2^2+\left(\frac{1}{2}-\frac{1}{4}\sqrt{2}\right)k_1k_2\right). \qquad (3.33)$$

Expression (3.33) approximates asymptotic stability boundary given by Eq.(3.30) in the vicinity of the point (3.29). Substituting formulae (3.32) into Eq.(2.24) and taking into account that

$$\frac{f_{k,2}}{f_{k,1}} = -(5\sqrt{2}+4), \quad \omega_0(h_{k,1}f_{k,2} - h_{k,2}f_{k,1}) = -\frac{5\sqrt{2}}{32}, \quad \mathbf{f}_k^T\mathbf{G}_k^\dagger\mathbf{f}_k = f_{k,1}^2\frac{5+2\sqrt{2}}{4}$$

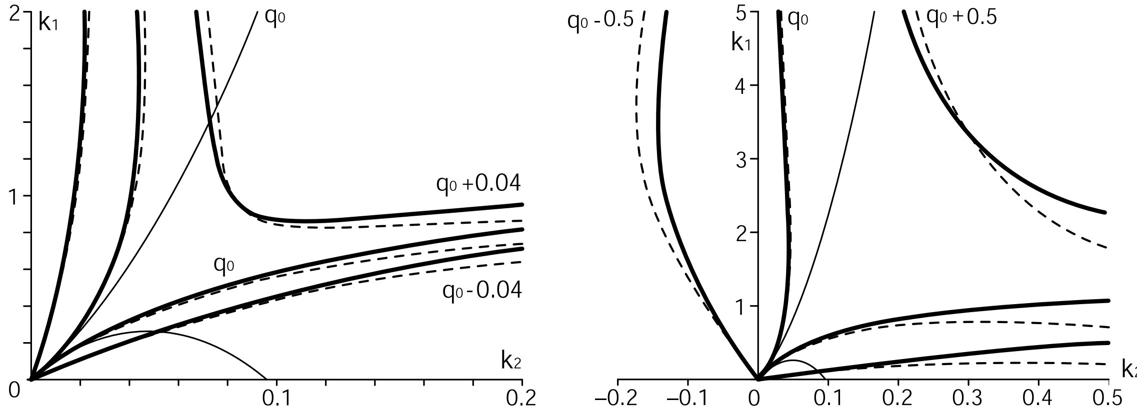

Figure 7. Stabilization domains for the Herrmann-Jong pendulum in the plane (k_1, k_2): Bold lines – exact solution (3.30), Dashed lines – approximation (3.33), Thin lines – approximation (3.34).

we find the approximation of the boundary of the stabilization domain in the plane of the parameters (k_1, k_2) for $q_{cr} = q_0$:

$$k_1 = (4 + 5\sqrt{2})k_2 \pm \sqrt{50(133 + 94\sqrt{2})}k_2^2 + O(k_2^3). \tag{3.34}$$

Asymptotics (3.34) of the stabilization domain were derived first in [19, 20] by the analysis of the Routh-Hurwitz inequalities for the Herrmann-Jong pendulum. Thus, we have obtained exactly the same result analyzing splitting of the double eigenvalue with a change of parameters.

The level curves of the boundary of the stabilization domain in the plane of the dissipation parameters k_1, k_2 given by Eq.(3.30), Eq.(3.33) and Eq.(3.34) are shown in Figure 7 by thick, dashed and thin lines, respectively. One can see that Eq.(3.33) gives more accurate approximation of the level curve for $q_{cr} = q_0$ than Eq.(3.34) because the latter contains only second order terms w.r.t. the parameter k_2.

Substitute $k_1{=}\delta\cos\alpha$, $k_2{=}\delta\sin\alpha$ in Eq.(3.30) and consider a limit as δ goes to zero. Then,

$$q_{cr}^{lim} = \frac{4 + 33\cos\alpha\sin\alpha}{12 + 14\cos\alpha\sin\alpha - 10\cos\alpha^2}. \tag{3.35}$$

Transforming Eq.(3.33) by the same way we get

$$q_{cr}^{lim} = \frac{314 - 144\sqrt{2} + (2\sqrt{2} + 168)\cos\alpha\sin\alpha + (140\sqrt{2} - 285)\cos\alpha^2}{((2 + \sqrt{2})\cos\alpha + (2 + 6\sqrt{2})\sin\alpha)^2}. \tag{3.36}$$

The graphs of function (3.35) and its approximation (3.36) are shown in Figure 8. One can see that the limit of the critical load depends on the direction in the plane of parameters, so the function $q_{cr}(k_1, k_2)$ has no limit when k_1, k_2 tend to zero.

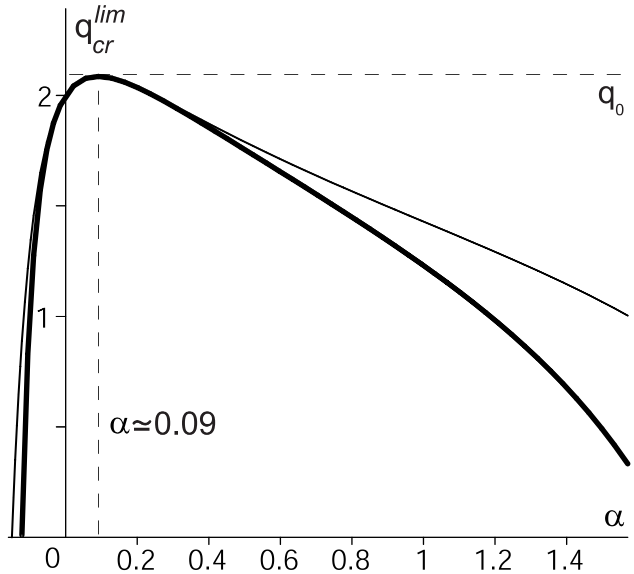

Figure 8. The critical load $q_{cr}^{lim}(\alpha)$ for the Herrmann-Jong pendulum at $\eta = 1$: Thick line – Eq.(3.35), Thin line – Eq.(3.36).

This fact was noted first in the work [20]. The function $q_{cr}^{lim}(\alpha) \le q_0$. This reflects a jump in the critical load due to small damping [3, 4]. However, for $k_1 = (4+5\sqrt{2})k_2$ (corresponding to $\alpha \simeq 0.09008$) a small dissipation stabilizes the pendulum, because in this case exact equation (3.30) as well as its approximation (3.33) take the form

$$q_{cr} = q_0 + \frac{k_2^2}{2}(4 + 5\sqrt{2}). \tag{3.37}$$

One can see that q_{cr} is a function of one of damping parameters and goes to q_0 as $k_2 \to 0$. Both curves given by Eqs.(3.35), (3.36) have a maximum $q_{cr}^{lim} = q_0$ exactly at the same point $\alpha = \arctan(1/(4+5\sqrt{2}))$ as shown in Figure 8. Therefore, equation (3.36) allows one to evaluate a jump in the critical load due to small damping with a good accuracy for the directions from a narrow angle in the plane of parameters k_1 and k_2. However, Eq.(3.33) gives a good approximation of asymptotic stability boundary (3.30) in the vicinity of the point (3.29), as one can see in Figure 7.

4 Analysis of characteristic polynomial

It is interesting and practical to express the conditions of asymptotic stability of the linear circulatory system perturbed by the velocity-dependent forces

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{D}\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = 0 \tag{4.1}$$

directly in terms of the entries of the $m\times m$ matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{A}$. However, even in the case of systems with two degrees of freedom such a representation in the general case is not evident [14, 31].

It turns out that the characteristic polynomial $P(\lambda){=}\det(\lambda^2\mathbf{M}{+}\lambda\mathbf{D}{+}\mathbf{A})$ of system (4.1) has a compact form, which is convenient for the investigation of stability. According to Eq.(B7), for the systems with 2 degrees of freedom the polynomial written in such a form looks like

$$P(\lambda) = \det \mathbf{M}\lambda^4 + \mathrm{tr}(\mathbf{D}^\dagger\mathbf{M})\lambda^3 + (\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}) + \det \mathbf{D})\lambda^2 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{D})\lambda + \det \mathbf{A}, \quad (4.2)$$

where $\mathbf{D}^\dagger$ and $\mathbf{A}^\dagger$ are the matrices adjoint to $\mathbf{D}$ and $\mathbf{A}$ [29]

$$\mathbf{D}^\dagger = \left[\begin{array}{cc} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{array}\right], \quad \mathbf{A}^\dagger = \left[\begin{array}{cc} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{array}\right].$$

In the case of arbitrary degrees of freedom the powerful tool allowing to represent a characteristic polynomial by means of the invariants of the matrices involved is the Leverrier–Faddejev algorithm, see Appendix B.

Stability of system (4.1) depends on the loci of the roots of the corresponding characteristic polynomial in the complex plane. In the work [16] the movement of eigenvalues of a conservative system due to addition of small gyroscopic forces was investigated with the use of the sensitivity analysis of the roots of the characteristic polynomial written in the form (4.2). The study of an influence of a small dissipation on the stability of a circulatory system in the works [21, 22] was based on the same approach. In all these works the systems with two degrees of freedom were considered and sensitivities of simple roots were derived and taken into account.

In the works [14, 17] a problem of determining a structure of the matrix $\mathbf{D}$, which realizes the stable perturbation of a circulatory system was posed. The methods proposed in these works deal with the eigenvalue problems and use eigenvalues and eigenvectors. In this Section we will show that at least for the systems with two degrees of freedom the analysis of the characteristic polynomial written in the form (4.2) allows one to find the structure of the stabilizer $\mathbf{D}$ in an easy and straightforward way with a clear interpretation as well as to find the approximation of the domain of asymptotic stability.

The Routh-Hurwitz criterion and its interpretation. Consider non-conservative system (4.1) with $m{=}2$ degrees of freedom. Restrict our consideration to the case when $\mathbf{M}$ is the identity matrix. Then, polynomial (4.2) takes more simple form

$$P(\lambda) = \lambda^4 + \mathrm{tr}\mathbf{D}\lambda^3 + (\mathrm{tr}\mathbf{A} + \det \mathbf{D})\lambda^2 + (\mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D} - \mathrm{tr}(\mathbf{A}\mathbf{D}))\lambda + \det \mathbf{A}. \quad (4.3)$$

Assume that for $\mathbf{D} = 0$ the spectrum of the unperturbed circulatory system

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{A}\mathbf{y} = 0 \tag{4.4}$$

consists of the double purely imaginary eigenvalue $\lambda_0 = i\omega_0$ with the Jordan chain of length 2. The necessary and sufficient condition for λ_0 to be double is

$$\det \mathbf{A} = \left(\frac{\mathrm{tr}\mathbf{A}}{2}\right)^2. \tag{4.5}$$

In this case

$$\omega_0^2 = \frac{\mathrm{tr}\mathbf{A}}{2}. \tag{4.6}$$

Circulatory system (4.4) belongs therefore to the boundary between the stability and flutter domains [26] and is unstable. How should one choose the matrix $\mathbf{D}$ of the velocity-dependent forces to make circulatory system (4.4) asymptotically stable?

Applying the Routh-Hurwitz criterion of asymptotic stability to polynomial (4.3) and taking into account conditions (4.5), (4.6) we find the inequalities describing the asymptotic stability domain in the vicinity of the circulatory system (4.4)

$$\begin{gathered}\mathrm{tr}\mathbf{D}>0, \quad 2\omega_0^2 + \det \mathbf{D}>0, \quad 2\omega_0^2\mathrm{tr}\mathbf{D}-\mathrm{tr}(\mathbf{AD})>0, \quad \omega_0^4 > 0,\\ \det \mathbf{D}\mathrm{tr}\mathbf{D}(2\omega_0^2\mathrm{tr}\mathbf{D}-\mathrm{tr}(\mathbf{AD}))>(\omega_0^2\mathrm{tr}\mathbf{D} - \mathrm{tr}(\mathbf{AD}))^2.\end{gathered} \tag{4.7}$$

One can see that inequalities (4.7) are equivalent to the following three conditions

$$\mathrm{tr}\mathbf{D}>0, \quad \det \mathbf{D}>0, \quad \det \mathbf{D}\mathrm{tr}\mathbf{D}(2\omega_0^2\mathrm{tr}\mathbf{D}-\mathrm{tr}(\mathbf{AD}))>(\omega_0^2\mathrm{tr}\mathbf{D} - \mathrm{tr}(\mathbf{AD}))^2. \tag{4.8}$$

Inequalities (4.8) are *the necessary and sufficient* conditions for the matrix of the velocity-dependent forces $\mathbf{D}$ to make circulatory system (4.4)–(4.6) with 2 degrees of freedom asymptotically stable.

One can see from Eqs.(4.8) that three parameters $\mathrm{tr}\mathbf{D}$, $\det \mathbf{D}$, and $\mathrm{tr}(\mathbf{AD})$ naturally appear in the stability conditions. The asymptotic stability boundary is therefore a surface

$$\det \mathbf{D}\mathrm{tr}\mathbf{D}(2\omega_0^2\mathrm{tr}\mathbf{D}-\mathrm{tr}(\mathbf{AD}))=(\omega_0^2\mathrm{tr}\mathbf{D} - \mathrm{tr}(\mathbf{AD}))^2, \quad \mathrm{tr}\mathbf{D}>0, \quad \det \mathbf{D}>0. \tag{4.9}$$

in the space of the parameters $\mathrm{tr}\mathbf{D}$, $\mathrm{tr}(\mathbf{AD})$, and $\det \mathbf{D}$, Figure 9.

Let us show that for small velocity-dependent forces this surface is the well-known Whitney umbrella [24]. For this purpose we assume $\mathbf{D} = \epsilon\widetilde{\mathbf{D}}$, where $\epsilon \geq 0$ is a small parameter. From equation (4.9) we can find that on the boundary of the asymptotic stability domain

$$\mathrm{tr}(\mathbf{AD}) = \mathrm{tr}\mathbf{D}\left(\omega_0^2 - \frac{\epsilon^2}{2}\det \widetilde{\mathbf{D}} \pm \epsilon\omega_0\sqrt{\det \widetilde{\mathbf{D}}}\sqrt{1+\epsilon^2\frac{\det \widetilde{\mathbf{D}}}{4\omega_0^2}}\right) =$$

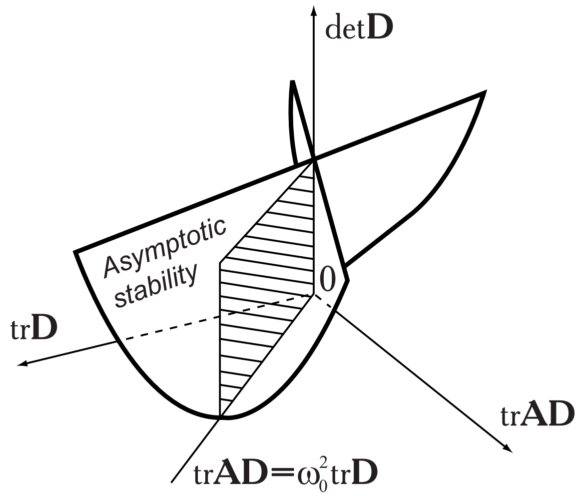

Figure 9. The geometrical meaning of the Routh-Hurwitz conditions (4.8): The Whitney umbrella (the tangent cone is hatched).

$$= \mathrm{tr}\mathbf{D}(\omega_0^2 \pm \epsilon\omega_0\sqrt{\det \widetilde{\mathbf{D}}} + O(\epsilon^2)). \tag{4.10}$$

Equation (4.10) without term $O(\epsilon^2)$ rewritten in the form

$$(\mathrm{tr}(\mathbf{AD}) - \omega_0^2 \mathrm{tr}\mathbf{D})^2 = \omega_0^2(\mathrm{tr}\mathbf{D})^2 \det \mathbf{D} \tag{4.11}$$

describes the surface $XY^2 = Z^2$ known as the Whitney umbrella [24] with $X = \det \mathbf{D}$, $Y = \omega_0 \mathrm{tr}\mathbf{D}$, $Z = \mathrm{tr}(\mathbf{AD}) - \omega_0^2\mathrm{tr}\mathbf{D}$.

Therefore, for fairly small perturbations $\mathbf{D}$ asymptotic stability domain (4.8) in the neighborhood of the circulatory system (4.4)–(4.6) is approximated by the following inequalities

$$\mathrm{tr}\mathbf{D} > 0, \quad \det \mathbf{D} > 0, \quad \omega_0^2(\mathrm{tr}\mathbf{D})^2 \det \mathbf{D} > (\mathrm{tr}(\mathbf{AD}) - \omega_0^2 \mathrm{tr}\mathbf{D})^2. \tag{4.12}$$

One can see that the asymptotic stability exists inside of a half of the Whitney umbrella, shown in Figure 9 in the space of the parameters $\mathrm{tr}\mathbf{D}$, $\det \mathbf{D}$, and $\mathrm{tr}(\mathbf{AD})$. At the origin the stability domain has a generic singularity "deadlock of an edge" [23, 24], corresponding to the double eigenvalue $i\omega_0$ of the matrix $\mathbf{A}$ of the unperturbed circulatory system (4.4).

Tangent cone to the stability domain and the structure of a stabilizer. Surface (4.9) has common points with the plane $\mathrm{tr}\mathbf{D}$, $\mathrm{tr}(\mathbf{AD})$ only along the line $\mathrm{tr}(\mathbf{AD}) = \omega_0^2\mathrm{tr}\mathbf{D}$. Two inequalities (4.9) distinguish the part of the plane containing this line and the vertical axis situated inside of the stability domain in a neighborhood of the origin. Thus, the tangent cone to the domain of the asymptotic stability at the origin (hatched in Figure 9), that is a set of direction vectors of the curves starting at this point and lying in the stability domain [23], is defined by the conditions

$$2\mathrm{tr}(\mathbf{AD}) = \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D}, \quad \det \mathbf{D} > 0, \quad \mathrm{tr}\mathbf{D} > 0. \tag{4.13}$$

Proposition: *Sufficient conditions for a small* 2×2 *matrix* $\mathbf{D}$ *to stabilize a circulatory system (4.4) with a given* 2×2 *matrix* $\mathbf{A}$ *satisfying Eq.(4.5) are Eqs.(4.13).*

Let us find the explicit form of the *symmetric* matrices realizing the perturbations stabilizing a circulatory system. Expressing the equality from (4.13) by means of the entries of the matrices $\mathbf{A}$ and $\mathbf{D}$ we get

$$(d_{22} - d_{11})(a_{22} - a_{11}) + 2(a_{12}d_{21} + a_{21}d_{12}) = 0, \quad d_{12} = d_{21}. \tag{4.14}$$

Isolating the term d_{12} in Eq.(4.14) we can write the structure of the matrix $\mathbf{D}$ as

$$\mathbf{D} = \begin{bmatrix} d_{11} & \frac{(a_{22}-a_{11})(d_{11}-d_{22})}{2(a_{12}+a_{21})} \\ \frac{(a_{22}-a_{11})(d_{11}-d_{22})}{2(a_{12}+a_{21})} & d_{22} \end{bmatrix}. \tag{4.15}$$

The structure (4.15) of the symmetric 2×2 matrices was found earlier in the work [14]. However, to stabilize a circulatory system the matrices $\mathbf{D}$ with such a structure must also satisfy the two inequalities (4.13). Calculating the determinant of matrix (4.15) and taking into account the positiveness of its trace we get the additional conditions on the entries of the symmetric matrix $\mathbf{D}$

$$d_{11},\ d_{22} > 0, \quad \frac{\sqrt{x}-1}{\sqrt{x}+1} < \frac{d_{11}}{d_{22}} < \frac{\sqrt{x}+1}{\sqrt{x}-1}, \quad x = 1 + \left(\frac{a_{22}-a_{11}}{a_{12}+a_{21}}\right)^2. \tag{4.16}$$

Therefore, symmetric matrices $\mathbf{D}$ with the structure given by Eqs.(4.15)–(4.16) realize stable perturbations of circulatory system (4.4), (4.5) with two degrees of freedom. Note that conditions (4.16) were not found in the work [14].

Sensitivity analysis of the roots of a characteristic polynomial. An analysis of the behaviour of eigenvalues of the characteristic polynomial gives an additional information on the stability and paves the way for a new interpretation.

Consider a linear autonomous non-conservative system with two degrees of freedom. Let the unperturbed circulatory system (4.4) have the double eigenvalue $\lambda_0 = i\omega_0$ with the Jordan chain of length 2. Perturbation of the circulatory system by the matrix $\epsilon\mathbf{D}$ gives the increment to the double eigenvalue:

$$\lambda(\epsilon) = i\omega_0 + \mu_1\sqrt{\epsilon} + \mu_2\epsilon + o(\epsilon). \tag{4.17}$$

The coefficients μ_1 and μ_2 are expressed by means of the derivatives of the characteristic polynomial $P(\lambda, \epsilon)$

$$P(\lambda,\epsilon) = \lambda^4 + \epsilon\mathrm{tr}\mathbf{D}\lambda^3 + (\mathrm{tr}\mathbf{A} + \epsilon^2\det\mathbf{D})\lambda^2 + \epsilon(\mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D} - \mathrm{tr}(\mathbf{AD}))\lambda + \det\mathbf{A}, \tag{4.18}$$

$$\mu_1^2 = -2\frac{\partial P}{\partial \epsilon}\left(\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}, \quad \mu_2 = \left(\frac{1}{3}\frac{\partial^3 P}{\partial \lambda^3}\frac{\partial P}{\partial \epsilon} - \frac{\partial^2 P}{\partial \lambda^2}\frac{\partial}{\partial \epsilon}\left(\frac{\partial P}{\partial \lambda}\right)\right)\left(\frac{\partial^2 P}{\partial \lambda^2}\right)^{-2}. \quad (4.19)$$

Therefore, the double root $i\omega_0$ of the polynomial $P(\lambda, \epsilon)$ splits with the variation of ϵ into two eigenvalues λ_1, λ_2 according to the formula [23]

$$\lambda_{1,2} = i\omega_0 \pm i\sqrt{\epsilon\frac{\partial P}{\partial \epsilon}\left(\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2}\right)^{-1}} + \epsilon\left(\frac{1}{3}\frac{\partial^3 P}{\partial \lambda^3}\frac{\partial P}{\partial \epsilon} - \frac{\partial^2 P}{\partial \lambda^2}\frac{\partial^2 P}{\partial \lambda \partial \epsilon}\right)\left(\frac{\partial^2 P}{\partial \lambda^2}\right)^{-2} + o(\epsilon), \quad (4.20)$$

if the radicand in Eq.(4.20) does not vanish. In the degenerate case when

$$\left.\frac{\partial P}{\partial \epsilon}\right|_{\epsilon=0,\ \lambda=i\omega_0} = 0 \quad (4.21)$$

the double eigenvalue splits linearly in ϵ:

$$\lambda(\epsilon) = i\omega_0 + \mu\epsilon + o(\epsilon), \quad (4.22)$$

where the coefficient μ is a solution of the following quadratic equation

$$\mu^2\frac{1}{2}\frac{\partial^2 P}{\partial \lambda^2} + \mu\frac{\partial^2 P}{\partial \lambda \partial \epsilon} + \frac{1}{2}\frac{\partial^2 P}{\partial \epsilon^2} = 0. \quad (4.23)$$

To derive Eq.(4.23) substitute Eq.(4.22) into the Taylor expansions of the function $P(\lambda, \epsilon)$, collect the terms with the same powers of ϵ, and equate them to zero. Then, the equation corresponding to ϵ is Eq.(4.21) and the equation corresponding to ϵ^2 is Eq.(4.23).

Let us now turn our attention to polynomial (4.18) and find how the double eigenvalue $i\omega_0$ splits along the stability boundary. Substituting the expression for $\mathrm{tr}\mathbf{AD}$ given by Eq.(4.10) into characteristic polynomial (4.18) we find that its first derivative with respect to ϵ becomes zero. Therefore, the splitting of the double eigenvalue $i\omega_0$ along the asymptotic stability boundary (4.9) is governed by Eqs.(4.22),(4.23).

Calculating the second derivatives of polynomial (4.18) under condition (4.10) and using them in Eq.(4.23) we obtain the quadratic equation on the coefficient μ

$$\mu^2 + \mu\frac{1}{2}\mathrm{tr}\mathbf{D} + \frac{1}{4}\det\mathbf{D} \pm i\frac{1}{4}\mathrm{tr}\mathbf{D}\sqrt{\det\mathbf{D}} = 0. \quad (4.24)$$

Calculating the roots of complex polynomial (4.24) with the use of Eqs.(C4) we find the expressions describing the splitting of the double eigenvalue $i\omega_0$

$$\lambda_1 = i\omega_0 \mp i\frac{\epsilon}{2}\sqrt{\det\mathbf{D}} + o(\epsilon), \quad \lambda_2 = i\omega_0 \pm i\frac{\epsilon}{2}\sqrt{\det\mathbf{D}} - \frac{\epsilon}{2}\mathrm{tr}\mathbf{D} + o(\epsilon). \quad (4.25)$$

For the complex-conjugate eigenvalue $-i\omega_0$ we have

$$\lambda_1' = -i\omega_0 \pm i\frac{\epsilon}{2}\sqrt{\det\mathbf{D}} + o(\epsilon), \quad \lambda_2' = -i\omega_0 \mp i\frac{\epsilon}{2}\sqrt{\det\mathbf{D}} - \frac{\epsilon}{2}\mathrm{tr}\mathbf{D} + o(\epsilon). \quad (4.26)$$

One can see from Eqs.(4.25), (4.26) that if simultaneously $\det\mathbf{D}>0$ and $\mathrm{tr}\mathbf{D}>0$, then the double purely imaginary eigenvalue splits on the stability boundary into purely imaginary eigenvalue and complex eigenvalue with negative real part. If these conditions are not valid, then complex eigenvalue with positive real part occurs as a result of the splitting. Thus, the above considerations give another (rather analytical than geometrical) explanation of asymptotic stability domain (4.8) as well as of sufficient conditions (4.13) for a small perturbation $\mathbf{D}$ to stabilize circulatory system (4.4), (4.5).

Example 1. We first consider the linear non-conservative system with 2 degrees of freedom with the generalized coordinates $y_1(t)$ and $y_2(t)$ [3, 7, 20]:

$$\frac{d^2y_1}{dt^2}+k_1\frac{dy_1}{dt}+\omega_1^2(y_1+qb_{12}y_2)=0,$$

$$\frac{d^2y_2}{dt^2}+k_2\frac{dy_2}{dt}+\omega_2^2(y_2+qb_{21}y_1)=0, \tag{4.27}$$

where ω_1 and ω_2 are the eigenfrequencies of a conservative system, k_1 and k_2 are the dissipation parameters, q is the non-conservative load parameter, and b_{12}, b_{21} are the coefficients of the matrix of non-conservative positional forces. It is assumed that $b_{12}b_{21}<0$ [20]. In this case

$$\mathbf{M}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{D}=\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}\omega_1^2 & \omega_1^2qb_{12}\\ \omega_2^2qb_{21} & \omega_2^2\end{bmatrix}. \tag{4.28}$$

The matrix $\mathbf{A}$ has the double eigenvalue $i\omega_0$ when the parameter q reaches its critical value

$$q_0=\frac{|\omega_2^2-\omega_1^2|}{2\omega_1\omega_2\sqrt{-b_{12}b_{21}}}. \tag{4.29}$$

If the dissipation parameters $k_1=0$, $k_2=0$ and the load parameter q is equal to q_0 given by Eq.(4.29), then system (4.27) is circulatory and belongs to the boundary between the stability and flutter domains. The matrix of the dissipative forces $\mathbf{D}$ in (4.28) is symmetric. If $k_1{=}k_2{>}0$, then according to Eqs.(4.15), (4.16) it realizes the stable perturbation of the initial circulatory system.

A more delicate result can be obtained after calculation of the quadratic approximation of the stability domain in the plane of parameters k_1, k_2. This domain defined by inequalities (4.8) has a boundary, which is a part of the surface (4.9) approximated by Eq.(4.11). All the ingredients needed to apply Eq.(4.11) are

$$\frac{\mathrm{tr}\mathbf{A}}{2}=\frac{\omega_1^2+\omega_2^2}{2}=\omega_0^2,\quad \mathrm{tr}\mathbf{D}=k_1+k_2,\quad \det\mathbf{D}=k_1k_2,\quad \mathrm{tr}\mathbf{AD}=k_1\omega_1^2+k_2\omega_2^2. \tag{4.30}$$

Substitution of Eqs.(4.30) into Eq.(4.11) yields

$$k_1\omega_1^2 + k_2\omega_2^2 = (k_1+k_2)\frac{\omega_1^2+\omega_2^2}{2} \pm \frac{\sqrt{\omega_1^2+\omega_2^2}}{\sqrt{2}}(k_1+k_2)\sqrt{k_1k_2}, \quad k_1, k_2 > 0. \quad (4.31)$$

Seeking for the coefficient k_1 in the form $k_1 = ak_2 + bk_2^2 + o(k_2^2)$ we get from Eq.(4.31)

$$a = 1, \quad b = \pm\frac{2\sqrt{2(\omega_1^2+\omega_2^2)}}{\omega_1^2-\omega_2^2} = \pm\frac{4\omega_0}{\omega_1^2-\omega_2^2}.$$

Finally, we arrive at the following equation, describing the stabilization domain

$$k_1 = k_2 \pm \frac{2\sqrt{2(\omega_1^2+\omega_2^2)}}{\omega_1^2-\omega_2^2}k_2^2 + o(k_2^2). \quad (4.32)$$

Approximation (4.32) exactly coincides with the equation of the stability boundary obtained earlier in the work [20] from the Routh-Hurwitz criterion applied directly to system (4.27).

Example 2. Consider again the Herrmann-Jong pendulum (3.1), (3.2) assuming $\eta = 1$. Since $\det \mathbf{M} \neq 0$ we can rewrite Eq.(4.1) in the following form

$$\frac{d^2\mathbf{y}}{dt^2} + \mathbf{M}^{-1}\mathbf{D}\frac{d\mathbf{y}}{dt} + \mathbf{M}^{-1}\mathbf{A}\mathbf{y} = 0, \quad (4.33)$$

where

$$\mathbf{M}^{-1}\mathbf{D} = \frac{1}{2}\begin{bmatrix} k_1+2k_2 & -2k_2 \\ -k_1-4k_2 & 4k_2 \end{bmatrix}, \quad \mathbf{M}^{-1}\mathbf{A} = \frac{1}{2}\begin{bmatrix} 3-q & q-2 \\ q-5 & 4-q \end{bmatrix}. \quad (4.34)$$

To find the approximation of the stability boundary (4.11) in the plane of parameters k_1 and k_2 we should just evaluate the invariants of the matrices at the point $q = 7/2 - \sqrt{2}$ corresponding to the double eigenvalue $i2^{-1/4}$:

$$\mathrm{tr}(\mathbf{M}^{-1}\mathbf{A}) = \sqrt{2}, \quad \mathrm{tr}(\mathbf{M}^{-1}\mathbf{D}) = \frac{1}{2}k_1 + 3k_2, \quad \det(\mathbf{M}^{-1}\mathbf{D}) = \frac{1}{2}k_1k_2,$$

$$\mathrm{tr}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M}^{-1}\mathbf{D}) = \left(-\frac{1}{2} + \frac{\sqrt{2}}{2}\right)k_1 + \left(-\frac{1}{2} + 3\sqrt{2}\right)k_2. \quad (4.35)$$

Now substitute Eqs.(4.35) into Eq.(4.11) and obtain

$$\left(-\frac{1}{2} + \frac{\sqrt{2}}{2}\right)k_1 + \left(-\frac{1}{2} + 3\sqrt{2}\right)k_2 = \frac{1}{\sqrt{2}}\left(\frac{1}{2}k_1 + 3k_2\right) \pm \sqrt{\frac{1}{\sqrt{2}}}\left(\frac{1}{2}k_1 + 3k_2\right)\sqrt{\frac{1}{2}k_1k_2}.$$

Looking for the coefficient k_1 in the form $k_1 = ak_2 + bk_2^2 + o(k_2^2)$ finally we get

$$k_1 = (5\sqrt{2}+4)k_2 \pm \sqrt{50(133+94\sqrt{2})}k_2^2 + o(k_2^2). \quad (4.36)$$

Stability boundaries (4.36) were first obtained in [19, 20] by the direct analysis of the characteristic equation of the Herrmann-Jong pendulum. It should also be noted that Eq.(4.36) completely coincides with Eq.(3.34) obtained by another approach based on the analysis of the eigenvalue problem.

Now we find the general structure of the symmetric matrices $\mathbf{M}^{-1}\mathbf{D}$ stabilizing the Herrmann-Jong pendulum (4.33), (4.34). Evaluating the coefficients in Eq.(4.15) we obtain

$$\mathbf{M}^{-1}\mathbf{D} = \begin{bmatrix} d_{11} & \frac{\sqrt{2}}{8}(d_{22} - d_{11}) \\ \frac{\sqrt{2}}{8}(d_{22} - d_{11}) & d_{22} \end{bmatrix}. \tag{4.37}$$

The entries of matrix (4.37) must satisfy the inequalities (4.16) that look here like

$$0.02944 \simeq 17 - 12\sqrt{12} < \frac{d_{11}}{d_{22}} < 17 + 12\sqrt{12} \simeq 33.9706. \tag{4.38}$$

In the work [14] the stabilizing symmetric matrix $\mathbf{M}^{-1}\mathbf{D}$ was found in the form

$$\mathbf{M}^{-1}\mathbf{D} = \begin{bmatrix} b_- & l_1 - l_2 \\ l_1 - l_2 & b_+ \end{bmatrix}, \quad b_\pm = 3(l_1 \pm l_2) \pm 2\sqrt{2}(l_1 - l_2), \quad l_1 + l_2 > 0. \tag{4.39}$$

One can verify that for $l_1 = l_2$ the characteristic equation is

$$\lambda^4 + 6l_1\lambda^3 + \sqrt{2}\lambda^2 + (3\sqrt{2} - 3/2)l_1\lambda + 1/2 = 0. \tag{4.40}$$

If we take, for example, $l_1 = 1$ in Eq.(4.39), then the complex eigenvalues with the positive real part

$$\lambda = 0.0109 \pm i0.67772. \tag{4.41}$$

are the roots of Eq.(4.40). This result is a consequence of the fact that the coefficients $b_\pm$ cited in (4.39) were found in [14] incorrectly. The correct coefficients are

$$b_\pm = 3(l_1 + l_2) \pm 2\sqrt{2}(l_2 - l_1). \tag{4.42}$$

In this case the characteristic polynomial corresponding to $l_1 = l_2$ has another form

$$\lambda^4 + 12\lambda^3 l_1 + (\sqrt{2} + 36l_1^2)\lambda^2 + 6\lambda l_1\sqrt{2} + 1/2 = 0. \tag{4.43}$$

For $l_1 = 1$ all the roots of polynomial (4.43) have negative real parts.

However, even with the correct coefficients (4.42) it is possible to find the combination of the parameters l_1 and l_2 ($l_1 + l_2 > 0$) so that matrix (4.39) will destabilize system (4.33). According to Eq.(4.38) we take

$$b_- = \frac{297 + 202\sqrt{2}}{297 - 202\sqrt{2}}\, b_+ \simeq 51.4324\, b_+.$$

In this case $l_1 = 100$, $l_2 = -1$, $l_1 + l_2 > 0$. The corresponding characteristic equation

$$\lambda^4 + 594\lambda^3 + (\sqrt{2} - 3600)\lambda^2 + 499\sqrt{2}\lambda + 1/2 = 0$$

has two negative and two positive real roots

$$\lambda_1 = -599.999,\ \ \lambda_2 = -0.00071,\ \ \lambda_3 = 0.20363,\ \ \lambda_4 = 5.79668.$$

Thus, the matrix that has a structure defined by Eq.(4.39) can destabilize system (4.33), (4.34). This fact shows the importance of conditions (4.38) (that are a particular case of conditions (4.13) and (4.16)) for the matrices of small velocity-dependent forces having a structure (4.37) to stabilize a circulatory system with two degrees of freedom. These conditions were missing in the work [14].

Conclusion

The problem of the influence of small forces dependent on velocity on the stability of linear autonomous non-conservative systems is considered. Two approaches based on the sensitivity analysis of eigenvalues and of the roots of the characteristic polynomial are developed and used for this purpose.

Explicit formulae describing splitting of multiple eigenvalues of a linear operator smoothly dependent on a spectral parameter and a vector of real parameters are derived both for the generic case and for some degenerations.

With their use the explicit formulae approximating the domain of asymptotic stability for the non-conservative system with m degrees of freedom dependent on parameters corresponding to small velocity-dependent forces and non-conservative positional forces are constructed. Approximations of the stabilization domain in the space of parameters corresponding to small velocity-dependent forces are found. These formulae allow one to calculate approximately the jump in the critical flutter load due to addition of small velocity-dependent forces.

The explicit asymptotic formulae describing behaviour of eigenvalues in the complex plane due to action of velocity-dependent forces and positional non-conservative loads are derived. With their use the change in the static and flutter instability mechanisms due to small velocity-dependent forces is investigated.

With the use of the Leverrier–Faddejev algorithm the explicit expression for the characteristic polynomial of an $m \times m$ matrix by means of its invariants is obtained. Such a representation is found also for the characteristic polynomial of a quadratic matrix pencil.

These results are applied for the detailed investigation of the stability of general non-conservative systems with 2 degrees of freedom. The necessary and sufficient conditions of asymptotic stability are obtained in terms of the matrices of the system. The geometrical interpretation of these conditions is given. The tangent cone to the asymptotic stability boundary giving simple and practical sufficient conditions of asymptotic stability is found. The structure of a stabilizing velocity-dependent perturbation of a circulatory system is established. The approximations of the asymptotic stability domain are obtained.

The developed theory is compared with the results of earlier investigations and used for the study of the classical mechanical problems by H. Ziegler, V.V. Bolotin and G. Herrmann. The examples considered show the applicability and accuracy of the theoretical results obtained in the present paper.

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Appendix A

Bifurcation of multiple eigenvalues of a linear operator. Consider an eigenvalue problem for a linear matrix operator $\mathbf{L}$ whose coefficients smoothly depend on a complex spectral parameter λ and a vector $\mathbf{p} \in R^n$

$$\mathbf{L}(\lambda, \mathbf{p})\mathbf{u} = 0. \tag{A1}$$

At a fixed vector $\mathbf{p}$ a value λ_0 of the spectral parameter, at which there exists a nontrivial solution $\mathbf{u}_0$ of Eq.(A1), is called *eigenvalue* whereas the vector $\mathbf{u}_0$ is called *eigenvector* of the operator $\mathbf{L}$ at the eigenvalue λ_0.

Suppose that at $\mathbf{p} = \mathbf{p}_0$ there exists a k-fold eigenvalue λ_0, which possesses a Jordan chain of vectors consisting of only one eigenvector $\mathbf{u}_0$ and associated vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$. Denote $\mathbf{L}_0 = \mathbf{L}(\lambda_0, \mathbf{p}_0)$. The vectors of the Jordan chain satisfy the equations [32]

$$\mathbf{L}_0\mathbf{u}_0 = 0, \quad \mathbf{L}_0\mathbf{u}_s = -\sum_{r=1}^{s} \frac{1}{r!}\frac{\partial^r \mathbf{L}}{\partial \lambda^r}\mathbf{u}_{s-r}, \quad s = 1, \ldots, k-1, \tag{A2}$$

where partial derivatives are evaluated at $\lambda = \lambda_0$.

The Jordan chain at the complex-conjugate eigenvalue $\overline{\lambda}_0$ of the operator $\mathbf{L}^*$ adjoint in the sense of Hermite to $\mathbf{L}$ is defined by the equations [32]

$$\mathbf{L}_0^*\mathbf{v}_0 = 0, \quad \mathbf{L}_0^*\mathbf{v}_s = -\sum_{r=1}^{s} \frac{1}{r!}\frac{\partial^r \mathbf{L}^*}{\partial \overline{\lambda}^r}\mathbf{v}_{s-r}, \quad s = 1, \ldots, k-1. \tag{A3}$$

Let us take a variation of the vector of parameters

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon\dot{\mathbf{p}} + \frac{\epsilon^2}{2}\ddot{\mathbf{p}} + o(\epsilon^2), \quad \epsilon \geq 0, \tag{A4}$$

where dot indicates differentiation with respect to the small parameter ϵ and all the derivatives are evaluated at $\epsilon = 0$. Then, $\mathbf{L}(\lambda, \mathbf{p}(\epsilon))$ can be represented in the form of Taylor expansion

$$\mathbf{L}(\lambda, \mathbf{p}(\epsilon)) = \sum_{r=0}^{\infty} \frac{(\lambda-\lambda_0)^r}{r!}\frac{\partial^r}{\partial \lambda^r}(\mathbf{L}+\epsilon\mathbf{L}_1+\epsilon^2\mathbf{L}_2) + o(\epsilon^2), \tag{A5}$$

where

$$\mathbf{L}_1 = \sum_{s=1}^{n} \frac{\partial \mathbf{L}}{\partial p_s}\dot{p}_s, \quad \mathbf{L}_2 = \frac{1}{2}\sum_{s=1}^{n} \frac{\partial \mathbf{L}}{\partial p_s}\ddot{p}_s + \frac{1}{2}\sum_{s,t=1}^{n} \frac{\partial^2 \mathbf{L}}{\partial p_s \partial p_t}\dot{p}_s\dot{p}_t, \tag{A6}$$

and all the derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$, $\lambda = \lambda_0$.

The perturbed eigenvalue $\lambda(\epsilon)$ and eigenvector $\mathbf{u}(\epsilon)$ are expressed by means of the Newton-Puiseux series [27]

$$\lambda = \lambda_0 + \lambda_1\epsilon^{1/k} + \lambda_2\epsilon^{2/k} + \ldots + \lambda_{k-1}\epsilon^{(k-1)/k} + \lambda_k\epsilon + \ldots, \tag{A7}$$

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{w}_1\epsilon^{1/k} + \mathbf{w}_2\epsilon^{2/k} + \ldots + \mathbf{w}_{k-1}\epsilon^{(k-1)/k} + \mathbf{w}_k\epsilon + \ldots \tag{A8}$$

Substitute expansions (A5)–(A8) into eigenvalue problem (A1) and collect the terms with the same powers of the small parameter ϵ. Then, the first k equalities are

$$\mathbf{L}_0\mathbf{w}_r = -\sum_{j=0}^{r-1}\left(\sum_{s=1}^{r-j}\frac{1}{s!}\frac{\partial^s\mathbf{L}}{\partial\lambda^s}\sum_{|\alpha|_s=r-j}\lambda_{\alpha_1}\cdots\lambda_{\alpha_s}\right)\mathbf{w}_j,\ \ r=1,\ldots,k-1; \tag{A9}$$

$$\mathbf{L}_0\mathbf{w}_k = -\mathbf{L}_1\mathbf{w}_0 - \sum_{j=0}^{k-1}\left(\sum_{s=1}^{k-j}\frac{1}{s!}\frac{\partial^s\mathbf{L}}{\partial\lambda^s}\sum_{|\alpha|_s=k-j}\lambda_{\alpha_1}\cdots\lambda_{\alpha_s}\right)\mathbf{w}_j,\ \ |\alpha|_s=\alpha_1+\ldots+\alpha_s, \tag{A10}$$

where $\mathbf{w}_0$ is equal to $\mathbf{u}_0$ and the indexes α_1, ..., α_{k-1} are positive integers.

Comparison of Eqs.(A9) with equations of the Jordan chain (A2) gives coefficients $\mathbf{w}_r$ in expansions (A8)

$$\mathbf{w}_r = \sum_{j=1}^{r}\mathbf{u}_j\sum_{|\alpha|_j=r}\lambda_{\alpha_1}\cdots\lambda_{\alpha_j},\ \ r=1,\ldots,k-1. \tag{A11}$$

With the use of vectors (A11) transform Eq.(A10) as follows

$$\mathbf{L}_0\mathbf{w}_k = -\mathbf{L}_1\mathbf{u}_0 - \lambda_1^k\sum_{r=1}^{k}\frac{1}{r!}\frac{\partial^r\mathbf{L}}{\partial\lambda^r}\mathbf{u}_{k-r} + \sum_{j=1}^{k-1}\mathbf{L}_0\mathbf{u}_j\sum_{|\alpha|_j=k}\lambda_{\alpha_1}\cdots\lambda_{\alpha_j}. \tag{A12}$$

Multiplying Eq.(A12) by the left eigenvector $\mathbf{v}_0$ and taking into account that

$$\mathbf{v}_0^T\mathbf{L}_0\mathbf{u}_1 = \ldots = \mathbf{v}_0^T\mathbf{L}_0\mathbf{u}_{k-1} = 0 \tag{A13}$$

we get the coefficient λ_1 in expansions (A7)

$$\lambda_1^k = -\mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_0\left(\sum_{r=1}^{k}\frac{1}{r!}\mathbf{v}_0^T\frac{\partial^r\mathbf{L}}{\partial\lambda^r}\mathbf{u}_{k-r}\right)^{-1}. \tag{A14}$$

Bifurcation of a double eigenvalue of a linear operator. Consider the case of the double eigenvalue λ_0 with the Jordan chain of length 2 in detail.

After perturbation (A4) eigenvalue $\lambda(\epsilon)$ and eigenvector $\mathbf{u}(\epsilon)$ are expressed by the Newton-Puiseux series (A7), (A8) with $k = 2$. Substituting these expansions along with Eqs.(A5), (A6) into eigenvalue problem (A1) and collecting the terms with the same powers of ϵ we get the equations

$$\mathbf{L}_0\mathbf{w}_1 = -\lambda_1\mathbf{L}'\mathbf{u}_0, \tag{A15}$$

$$\mathbf{L}_0\mathbf{w}_2 = -\lambda_1\mathbf{L}'\mathbf{w}_1 - \lambda_2\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_1\mathbf{u}_0 - \frac{\lambda_1^2}{2!}\mathbf{L}''\mathbf{u}_0, \tag{A16}$$

$$\mathbf{L}_0\mathbf{w}_3 = -\lambda_1\mathbf{L}'\mathbf{w}_2 - \lambda_2\mathbf{L}'\mathbf{w}_1 - \lambda_3\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_1\mathbf{w}_1 - \lambda_1\mathbf{L}_1'\mathbf{u}_0 - \frac{\lambda_1^2}{2!}\mathbf{L}''\mathbf{w}_1 - \lambda_1\lambda_2\mathbf{L}''\mathbf{u}_0 - \frac{\lambda_1^3}{3!}\mathbf{L}'''\mathbf{u}_0, \tag{A17}$$

$$\mathbf{L}_0\mathbf{w}_4 = -\lambda_3\mathbf{L}'\mathbf{w}_1 - \lambda_2\mathbf{L}'\mathbf{w}_2 - \mathbf{L}_1\mathbf{w}_2 - \lambda_2\mathbf{L}'_1\mathbf{u}_0 - \frac{1}{2}\lambda_2^2\mathbf{L}''\mathbf{u}_0 - \lambda_4\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_2\mathbf{u}_0 -$$
$$-\lambda_1(\mathbf{L}'\mathbf{w}_3 + \lambda_2\mathbf{L}''\mathbf{w}_1 + \mathbf{L}_1'\mathbf{w}_1 + \lambda_3\mathbf{L}''\mathbf{u}_0) - \frac{\lambda_1^2}{2!}(\mathbf{L}''\mathbf{w}_2 + \lambda_2\mathbf{L}'''\mathbf{u}_0 + \mathbf{L}_1''\mathbf{u}_0) -$$
$$-\frac{\lambda_1^3}{3!}\mathbf{L}'''\mathbf{w}_1 - \frac{\lambda_1^4}{4!}\mathbf{L}''''\mathbf{u}_0, \tag{A18}$$

where prime ($' = \partial/\partial\lambda$) indicates a partial derivative with respect to the spectral parameter. Note that equation (A15) follows from Eq.(A9) with $r = 1$ and equation (A16) is a particular case of Eq.(A10) with $k = 2$. Thus, the coefficient λ_1 in expansion (A7) is given by formula (A14) where one should take $k = 2$

$$\lambda_1^2 = -\frac{\mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_0}{\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1 + \frac{1}{2}\mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}. \tag{A19}$$

From Eq.(A15) it follows that $\mathbf{w}_1 = \lambda_1\mathbf{u}_1 + \gamma\mathbf{u}_0$. To get the coefficient λ_2 one needs first to substitute the vector $\mathbf{w}_1$ in the explicit form into Eqs.(A16) and (A17). Then multiply Eq.(A16) by the left associated vector $\mathbf{v}_1$ to find the quantity $\mathbf{v}_0^T\mathbf{L}'\mathbf{w}_2$. Finally, multiply Eq.(A17) by the left eigenvector $\mathbf{v}_0$, substitute the term $\mathbf{v}_0^T\mathbf{L}'\mathbf{w}_2$ into the expression obtained, and isolate the coefficient λ_2

$$\lambda_2 = -\frac{\mathbf{v}_1^T\mathbf{L}_1\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}_1'\mathbf{u}_0 + \lambda_1^2 Q}{2\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}, \tag{A20}$$

$$Q = \mathbf{v}_1^T\mathbf{L}'\mathbf{u}_1 + \frac{1}{2!}(\mathbf{v}_1^T\mathbf{L}''\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{L}''\mathbf{u}_1) + \frac{1}{3!}\mathbf{v}_0^T\mathbf{L}'''\mathbf{u}_0.$$

Choosing the vectors $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{u}_1$, $\mathbf{v}_1$ so that $Q=0$ we rewrite Eq.(A20) in the form

$$\lambda_2 = -\frac{\mathbf{v}_1^T\mathbf{L}_1\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{L}_1\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}_1'\mathbf{u}_0}{2\mathbf{v}_0^T\mathbf{L}'\mathbf{u}_1 + \mathbf{v}_0^T\mathbf{L}''\mathbf{u}_0}. \tag{A21}$$

Therefore, the double eigenvalue λ_0 of the linear matrix operator $\mathbf{L}(\lambda, \mathbf{p}(\epsilon))$ splits in the case of general position according to the formula

$$\lambda(\epsilon) = \lambda_0 + \lambda_1\epsilon^{1/2} + \lambda_2\epsilon + o(\epsilon), \tag{A22}$$

with the coefficients λ_1 and λ_2 from Eqs.(A19), (A21).

The case when the coefficient λ_1 disappears in Eq.(A22) is degenerate and should be investigated separately. Substituting $\lambda_1=0$ into Eqs.(A15)–(A18) we find

$$\mathbf{L}_0\mathbf{w}_1 = 0, \quad \mathbf{L}_0\mathbf{w}_2 = -\lambda_2\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_1\mathbf{u}_0, \tag{A23}$$

$$\mathbf{L}_0\mathbf{w}_4 = -\lambda_3\mathbf{L}'\mathbf{w}_1 - \lambda_2\mathbf{L}'\mathbf{w}_2 - \mathbf{L}_1\mathbf{w}_2 - \lambda_2\mathbf{L}'_1\mathbf{u}_0 - \frac{1}{2}\lambda_2^2\mathbf{L}''\mathbf{u}_0 - \lambda_4\mathbf{L}'\mathbf{u}_0 - \mathbf{L}_2\mathbf{u}_0. \tag{A24}$$

Solving Eqs.(A23) find the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$

$$\mathbf{w}_1 = \beta \mathbf{u}_0, \quad \mathbf{w}_2 = \lambda_2 \mathbf{u}_1 + \gamma \mathbf{u}_0 - \mathbf{G}_0(\mathbf{L}_1 \mathbf{u}_0), \tag{A25}$$

where $\mathbf{G}_0$ is the operator inverse to $\mathbf{L}_0$. Multiply Eq.(A24) by the left eigenvector $\mathbf{v}_0$. Then, substitute in the result the quantity $\mathbf{v}_0^T \mathbf{L}' \mathbf{w}_2$ obtained from the multiplication of the second of equations (A23) by the left associated vector $\mathbf{v}_1$. After this transformation substitute the vectors (A25) into Eq.(A24). Finally, we arrive at the quadratic equation serving for the determination of the coefficient λ_2 in the degenerate case

$$\lambda_2^2 + \lambda_2 \frac{\mathbf{v}_1^T \mathbf{L}_1 \mathbf{u}_0 + \mathbf{v}_0^T \mathbf{L}_1 \mathbf{u}_1 + \mathbf{v}_0^T \mathbf{L}_1' \mathbf{u}_0}{\mathbf{v}_0^T \mathbf{L}' \mathbf{u}_1 + \frac{1}{2} \mathbf{v}_0^T \mathbf{L}'' \mathbf{u}_0} + \frac{\mathbf{v}_0^T \mathbf{L}_2 \mathbf{u}_0 - \mathbf{v}_0^T \mathbf{L}_1 \mathbf{G}_0(\mathbf{L}_1 \mathbf{u}_0)}{\mathbf{v}_0^T \mathbf{L}' \mathbf{u}_1 + \frac{1}{2} \mathbf{v}_0^T \mathbf{L}'' \mathbf{u}_0} = 0. \tag{A26}$$

The obtained formulae generalize the results on bifurcation of eigenvalues derived earlier in [17, 18, 19, 23, 26].

Appendix B

Representation of the characteristic polynomial of a matrix through its invariants. Consider the characteristic polynomial of a matrix $\mathbf{N} \in R^{m \times m}$

$$P_N(\lambda) = \det(\mathbf{N} - \lambda \mathbf{I}) = \sum_{r=0}^{m} p_r \lambda^{m-r}, \tag{B1}$$

where $\mathbf{I} \in R^{m \times m}$ is the identity matrix. The coefficients p_r can be expressed through the invariants of the matrix $\mathbf{N}$ according to the *Leverrier-Faddejev algorithm* [29, 33]

$$p_0 = 1, \; p_r = -\frac{1}{r} \mathrm{tr}(\mathbf{N}\mathbf{N}_{r-1}); \; \mathbf{N}_0 = \mathbf{I}, \; \mathbf{N}_r = \mathbf{N}\mathbf{N}_{r-1} + p_r \mathbf{I}, \; r = 1 \ldots m. \tag{B2}$$

For example,

$$P_N(\lambda) = \lambda^m - \frac{1}{1!}[\mathrm{tr}\mathbf{N}]\lambda^{m-1} + \frac{1}{2!}\left((\mathrm{tr}\mathbf{N})^2 - \mathrm{tr}\mathbf{N}^2\right)\lambda^{m-2} -$$
$$-\frac{1}{3!}\left((\mathrm{tr}\mathbf{N})^3 - 3\mathrm{tr}\mathbf{N}\mathrm{tr}\mathbf{N}^2 + 2\mathrm{tr}\mathbf{N}^3\right)\lambda^{m-3} +$$
$$+\frac{1}{4!}\left((\mathrm{tr}\mathbf{N})^4 - 6(\mathrm{tr}\mathbf{N})^2\mathrm{tr}\mathbf{N}^2 + 3(\mathrm{tr}\mathbf{N}^2)^2 + 8\mathrm{tr}\mathbf{N}\mathrm{tr}\mathbf{N}^3 - 6\mathrm{tr}\mathbf{N}^4\right)\lambda^{m-4} + \ldots,$$

where the numerical coefficients at the traces are called *multinomial* coefficients [34]. With the use of the notation from [34] we get the explicit expression for the characteristic polynomial

$$P_N(\lambda) = \lambda^m + \sum_{k=1}^{m} \frac{\lambda^{m-k}}{k!} \sum_{\|\alpha\|_k = k} \frac{k!}{\alpha_1! \cdots \alpha_k!} \left(-\frac{\mathrm{tr}\mathbf{N}^1}{1}\right)^{\alpha_1} \cdots \left(-\frac{\mathrm{tr}\mathbf{N}^k}{k}\right)^{\alpha_k},$$

$$\|\alpha\|_k = \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k. \tag{B3}$$

The indexes α_j are non-negative integers. Formula (B3) can be proven by induction taking into account that $\mathbf{N}\mathbf{N}_r = \mathbf{N}^{r+1} + \sum_{k=1}^{r} p_k \mathbf{N}^{r-k+1}$. Since $P_N(0) = \det \mathbf{N}$, the following relation is true for an $m \times m$ matrix $\mathbf{N}$

$$\sum_{\alpha_1+2\alpha_2+\ldots+k\alpha_k=k} \frac{(-\mathrm{tr}\mathbf{N}^1)^{\alpha_1} \cdots \left(-\mathrm{tr}\mathbf{N}^k\right)^{\alpha_k}}{1^{\alpha_1}\alpha_1! \cdots k^{\alpha_k}\alpha_k!} = \begin{cases} \det \mathbf{N}, & k = m \\ 0, & k > m \end{cases} \tag{B4}$$

Characteristic polynomial of a block matrix. Let now the matrix $\mathbf{N}$ be a 2×2 block matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A} & -\mathbf{D} \end{bmatrix}, \tag{B5}$$

where the matrix with the zero entries $\mathbf{0}$, the identity matrix $\mathbf{I}$, and $\mathbf{A}, \mathbf{D}$ are real $m \times m$ matrices. The characteristic polynomial of the matrix $\mathbf{N}$ is defined by the equation

$$P_N(\lambda) = \det(\lambda^2 + \mathbf{D}\lambda + \mathbf{A}).$$

Application of the algorithm (B2) to matrix (B5) gives

$$P_N(\lambda) = \det(\mathbf{A}+\lambda^2\mathbf{I}) + \lambda^m \det(\mathbf{D}+\lambda\mathbf{I}) - \lambda^{2m} +$$
$$+(\mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D} - \mathrm{tr}\mathbf{A}\mathbf{D})\lambda^{2m-3} + \frac{(\mathrm{tr}\mathbf{D})^2\mathrm{tr}\mathbf{A} - 2\mathrm{tr}\mathbf{A}\mathbf{D}\mathrm{tr}\mathbf{D} - \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D}^2 + 2\mathrm{tr}\mathbf{A}\mathbf{D}^2}{2}\lambda^{2m-4} + \ldots$$

In particular case, when $m = 2$, we have

$$P_N(\lambda) = \lambda^4 + \mathrm{tr}\mathbf{D}\lambda^3 + (\mathrm{tr}\mathbf{A} + \det \mathbf{D})\lambda^2 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{D})\lambda + \det \mathbf{A}, \tag{B6}$$

where the adjoint matrix $\mathbf{A}^\dagger$ is defined by the relation $\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}\det \mathbf{A}$, [29].

For the characteristic polynomial

$$P_1(\lambda) = \det(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{A})$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$ are 2×2 real matrices we obtain

$$P_1(\lambda) = \det \mathbf{M}\lambda^4 + \mathrm{tr}(\mathbf{D}^\dagger\mathbf{M})\lambda^3 + (\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}) + \det \mathbf{D})\lambda^2 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{D})\lambda + \det \mathbf{A}. \tag{B7}$$

The characteristic polynomial with the complex coefficients

$$P_2(\lambda) = \det(\mathbf{M}\lambda^2 + (\mathbf{D} - i\mathbf{G})\lambda + (\mathbf{K} - i\mathbf{N})) \tag{B8}$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{G}$, $\mathbf{K}$ and $\mathbf{N}$ are real $m \times m$ matrices and i is the imaginary unit usually appears in gyrodynamics [35]. On the basis of Eq.(B7) one can get the explicit form of the polynomial (B8) in the case when $m = 2$

$$P_2(\lambda) = \det \mathbf{M}\lambda^4 + (\mathrm{tr}\mathbf{D}^\dagger\mathbf{M} - i\mathrm{tr}\mathbf{G}^\dagger\mathbf{M})\lambda^3 +$$
$$+(\det \mathbf{D} - \det \mathbf{G} + \mathrm{tr}\mathbf{K}^\dagger\mathbf{M} - i(\mathrm{tr}\mathbf{N}^\dagger\mathbf{M} + \mathrm{tr}\mathbf{D}^\dagger\mathbf{G}))\lambda^2 +$$
$$+(\mathrm{tr}\mathbf{K}^\dagger\mathbf{D} - \mathrm{tr}\mathbf{N}^\dagger\mathbf{G} - i(\mathrm{tr}\mathbf{K}^\dagger\mathbf{G} + \mathrm{tr}\mathbf{N}^\dagger\mathbf{D}))\lambda + (\det \mathbf{K} - \det \mathbf{N} - i\mathrm{tr}\mathbf{K}^\dagger\mathbf{N}). \tag{B9}$$

Appendix C

Stable complex polynomials. Consider a quadratic polynomial with the complex coefficients

$$(\lambda - i\omega_0)^2 + (\lambda - i\omega_0)(a + ib) + (c + id). \tag{C1}$$

According to the Bilharz criterion [28] the roots of polynomial (C1) have negative real parts if and only if

$$c > \frac{d^2}{a^2} - b\frac{d}{a}, \ a > 0. \tag{C2}$$

The asymptotic stability boundary has the form

$$c = \frac{d^2}{a^2} - b\frac{d}{a}, \ a > 0. \tag{C3}$$

In this case the roots of polynomial (C1) are as follows

$$\lambda_1 = i\omega_0 - i\frac{d}{a}, \quad \lambda_2 = i\omega_0 + i\frac{d}{a} - a - ib. \tag{C4}$$